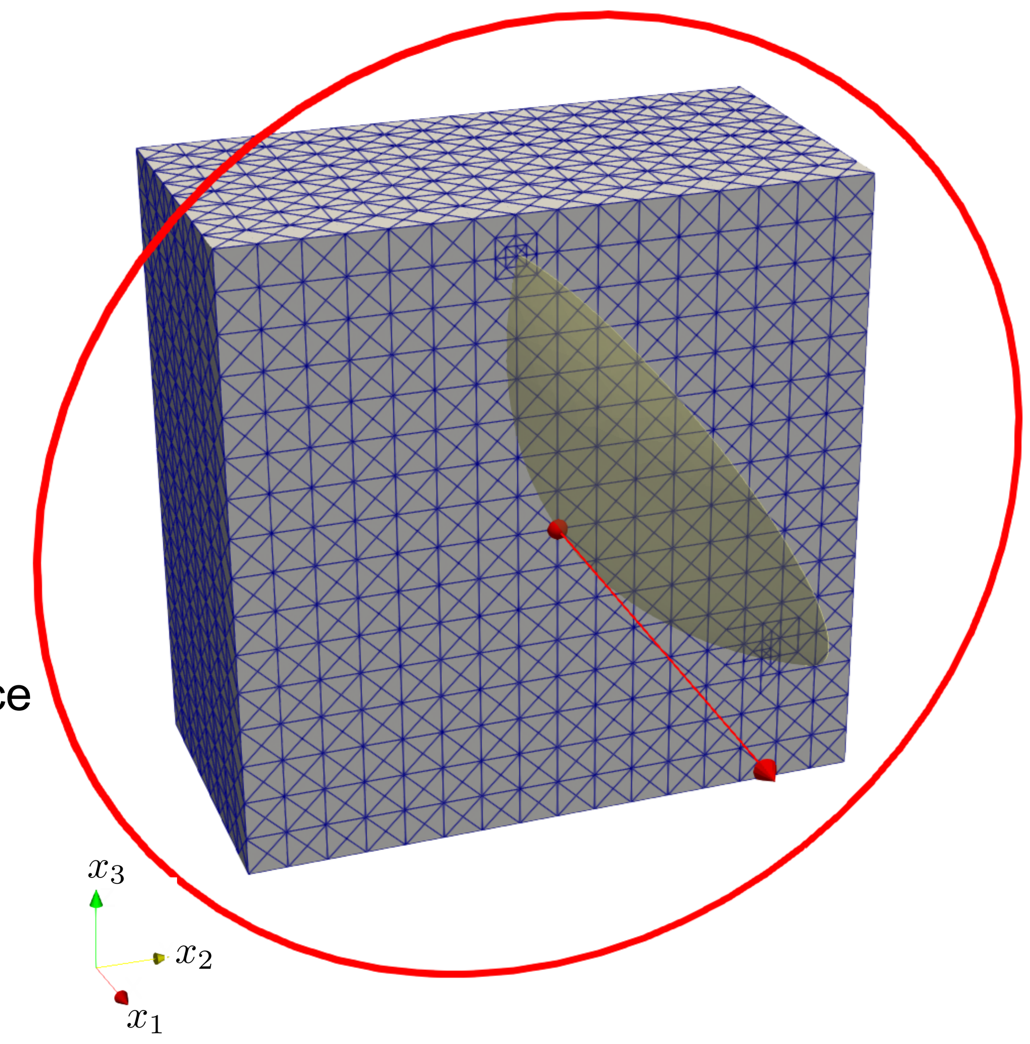
High-order formulations, a posteriori error estimators, and adaptive procedures for the G/XFEM

Murilo H. C. Bento[1], Sergio P. B. Proença[1], C. Armando Duarte[2]

[1] Department of Structural Engineering, São Carlos School of Engineering, University of São Paulo

[2] Newmark Laboratory, Department of Civil and Environmental Engineering, University of Illinois Urbana-Champaign

Introduction



Context

The G/XFEM can accurately and efficiently solve problems in which prior knowledge about their solutions is available. Typical classes of problems in this category are those from LEFM. The method has proven to deliver well-conditioned and optimally 1st-order convergent solutions. Nevertheless, the development of higher-order approximations is ongoing and is of interest due to their **higher convergence rates** and because 1st-order G/XFEM may be not competitive with 2nd-order FEM for 3-D LEFM problems. Also, when it comes to assessing the accuracy of G/XFEM approximations, special tools are needed when the exact solution is unknown and **a posteriori error estimators** play an important role in this. Besides being able to estimate well global discretization errors, these error estimators need also to be locally effective and, therefore, they can be readily applied to drive **adaptive simulations**. This type of simulation is important for 3-D LEFM problems since only singular enrichments and uniform mesh refinement are not enough for optimal convergence.

Main objective

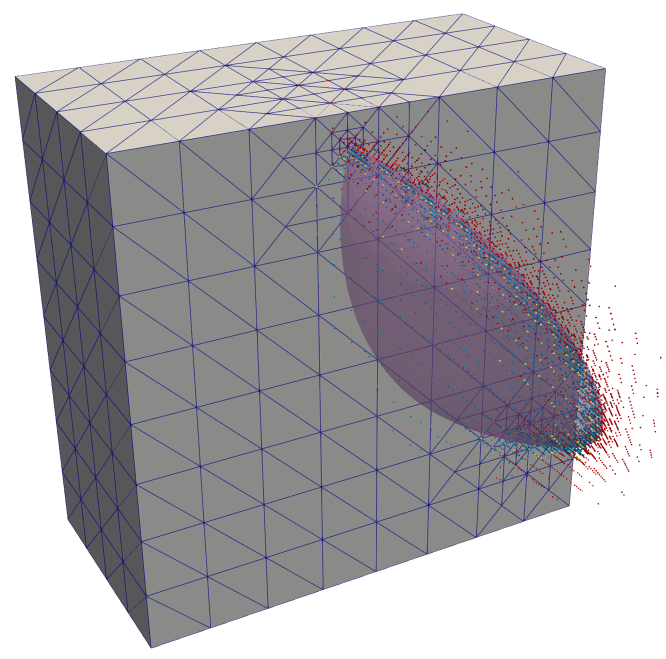
- **Development of h-adaptive procedures able to**
  1. Recover optimal 2nd-order convergence rates for 3-D LEFM problems.
  2. Deliver solutions with discretization errors smaller than a pre-specified tolerance.

2nd-order G/XFEM

We propose [1] a 2nd-order G/XFEM for 2- and 3-D LEFM problems that is well-conditioned and optimally convergent.

$$\mathcal{S}_{\mathrm{FEM^2\text{-}DSGFEM}} = \mathcal{S}_{\mathrm{FEM}} + \mathcal{S}^{D}_{\mathrm{ENR}} + \mathcal{S}^{S}_{\mathrm{ENR}}$$

Standard 2nd-order Lagrangian FEM



- 3-D LEFM problems still need mesh refinement to attain optimal convergence.

ZZ-BD error estimator

To come up with adaptive algorithms and assess the accuracy of numerical approximations in the cases the exact solution is unknown, we also propose [2] an accurate and efficient error estimator, tailored for 2nd-order approximations.

- **Recovery procedure**

$$\sigma^*(\mathbf{x}) = \sum_{\alpha \in \mathscr{I}_\times} \varphi_\alpha(\mathbf{x}) \sum_{i=0}^{n^R_\alpha} \boldsymbol{a}_{\alpha i} \odot \mathcal{R}_{\alpha i}(\mathbf{x})$$

- **Estimation of the discretization error**

  ϵ^* is estimated by comparing the numerical and recovered stress solutions.

h-Adaptive algorithm and numerical example

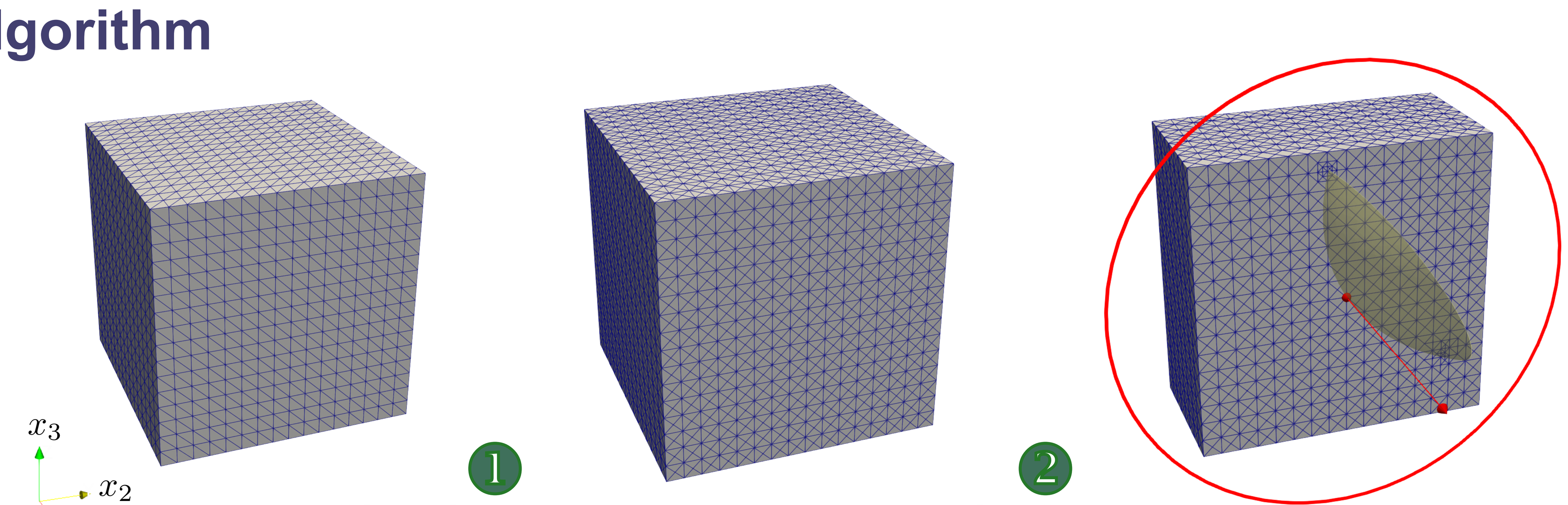
2nd-order G/XFEM needs **local mesh refinement** near crack fronts to attain optimal convergence in 3-D and the level of refinement that must be used is problem-dependent, difficult to be defined a priori, and usually relies on rules of thumb or trial-and-error estrategies.

Objective

An h-adaptive algorithm able to build on the fly generalized finite element discretizations that deliver optimal convergence rates and relative errors smaller than a pre-specified tolerance is developed.

- It is based on classical strategies, like the ones proposed by Zienkiewicz's and Oñate's researches [3, 4], but simplified and tailored to work well with 3-D LEFM problems.

Algorithm



Global refinement

A global refinement parameter is defined and used to compute how much all the elements in the mesh must be refined so $\epsilon_r \leq \bar{\epsilon}_r$.

Local refinement

A local refinement parameter is defined and computed for all the elements that are touched or crossed by the crack front. It is used to compute how much these elements must be refined.

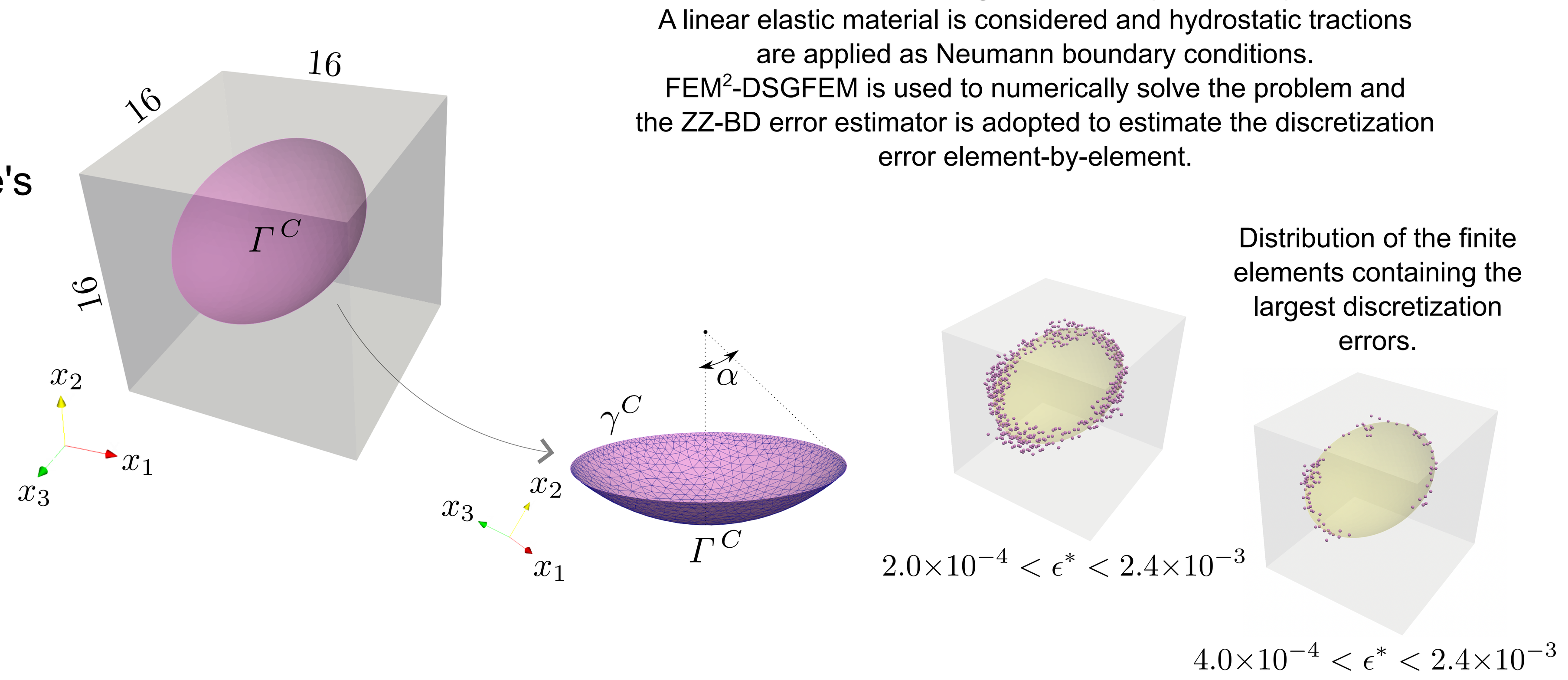
Obs.: Mesh generation - Gmsh + bisection refinement algorithm.

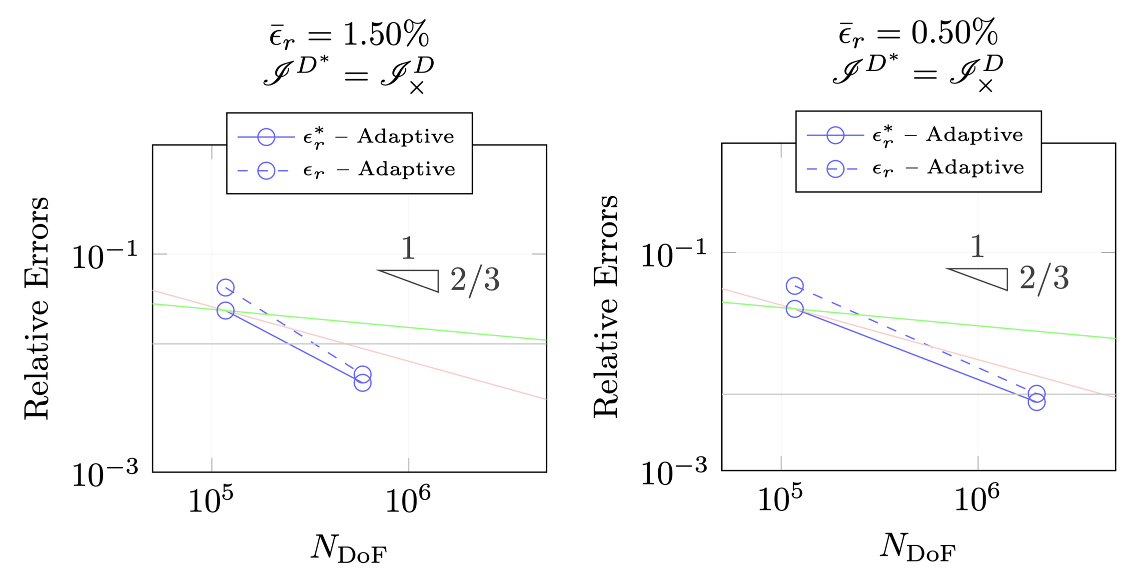
Assumption

The entire local error is caused by the crack singularity and it is expected that away from it the solution is smooth and the discretization error is low and nearly constant.

Numerical example

Cubic domain containing an inclined spherical cap crack. A linear elastic material is considered and hydrostatic tractions are applied as Neumann boundary conditions. FEM²-DSGFEM is used to numerically solve the problem and the ZZ-BD error estimator is adopted to estimate the discretization error element-by-element.





Convergence of relative errors for adaptive and standard FEM² and FEM²-DSGFEM simulations.

Conclusions

- The proposed h-adaptive strategy:
  - Is able to deliver on the fly discretizations that converge at the optimal rate of $\mathcal{O}(N_{\mathrm{DoF}}{}^{-2/3})$.
  - Is able to deliver final discretizations with error smaller than the pre-specified tolerance.
  - Requires very few steps to converge - in general, only 1.

Acknowledgments

FAPESP
SÃO PAULO RESEARCH FOUNDATION

SOARD
Southern Office of Aerospace Research and Development

References: [1] 10.1016/j.cma.2022.114917 [2] 10.1002/nme.7238 [3] 10.1002/nme.1620240206 [4] 10.1108/eb023910