The Generalized Finite Element Method as a Framework for Multiscale Structural Analysis

C. Armando Duarte
Dept. of Civil and Environmental Engineering
Computational Science and Engineering
University of Illinois at Urbana-Champaign, Urbana, IL

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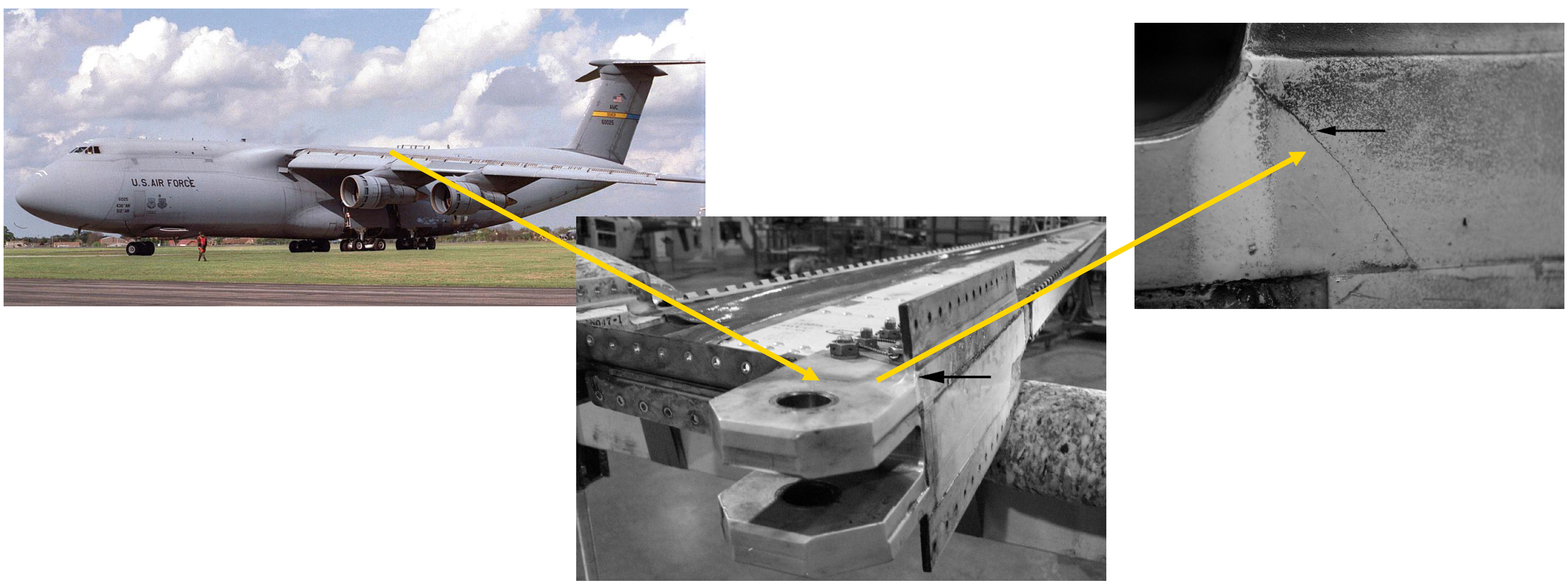
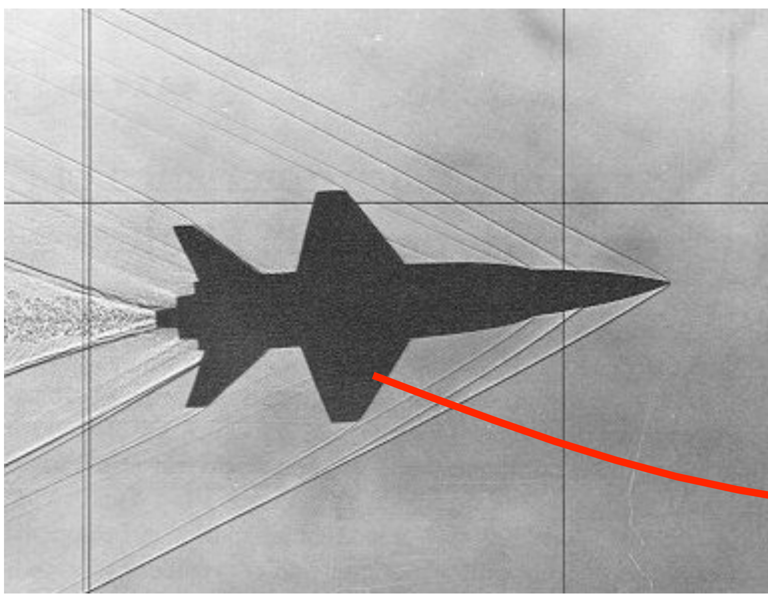
Motivation: US Air Force "Digital Twin"

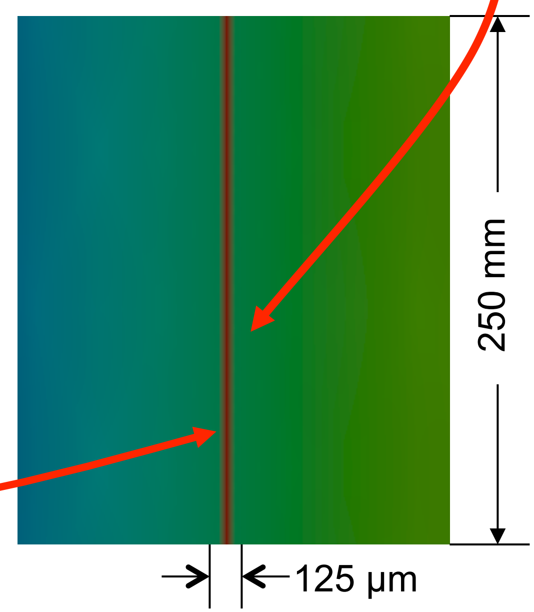
- A Digital Twin is a computational model of a *specific* aircraft
- The model will be flown through same flight profiles as recorder for the actual aircraft
- The digital model will be used to determine when and where structural damage is likely to occur
- *A Digital Twin must capture responses and interactions among a broad range of scales*: From aircraft scale to highly localized damage in one of its components



Bridging Scales

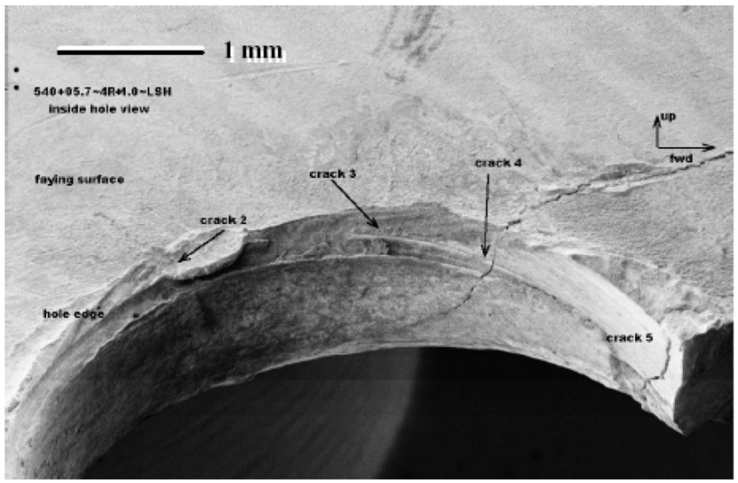
- **Thermal loads on hypersonic aircrafts**
- Shock wave impingements cause large thermal gradients
- Experiments are difficult and limited





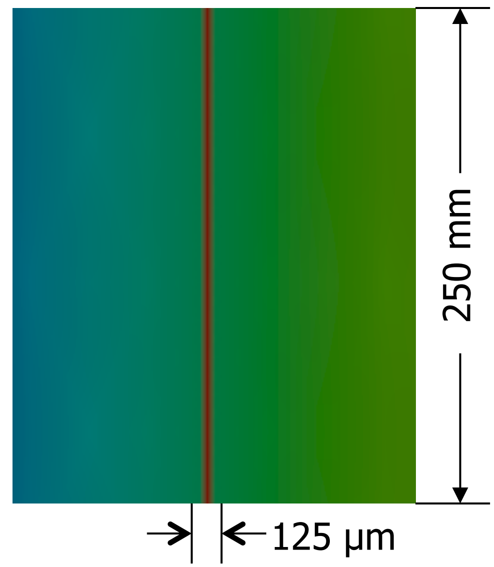
Dimensions not to scale

Multiscale Structural Problems



Multiple cracks around a rivet hole

[Sandia National Lab, 2005]



Thermal loads on hypersonic aircrafts
(dimensions not to scale)

- Predictive simulations require modeling of phenomena spanning several spatial and temporal scales
- Advances in existing computational methods are needed
- Increasing computational power alone is not enough

Outline

- Motivation for Multiscale Structural Analysis
- Bridging Scales with the GFEM:
  - Global-local enrichments
  - Verification
- Mathematical Analysis and Implementation
- Applications and Computational Efficiency
- Transition: Non-intrusive implementation in Abaqus
- Parallel Computation of Enrichment Functions
- Enrichment Functions for Confined Plasticity Problems

Early works on Generalized FEMs

- Babuska, Caloz and Osborn, 1994 (Special FEM).
- Duarte and Oden, 1995 (Hp Clouds).
- Babuska and Melenk, 1995 (PUFEM).
- Oden, Duarte and Zienkiewicz, 1996 (Hp Clouds/GFEM).
- Duarte, Babuska and Oden, 1998 (GFEM).
- Belytschko et al., 1999 (Extended FEM).
- Strouboulis, Babuska and Copps, 2000 (GFEM).

- Basic idea:
  - Use a partition of unity to build Finite Element shape functions

- Recent review paper

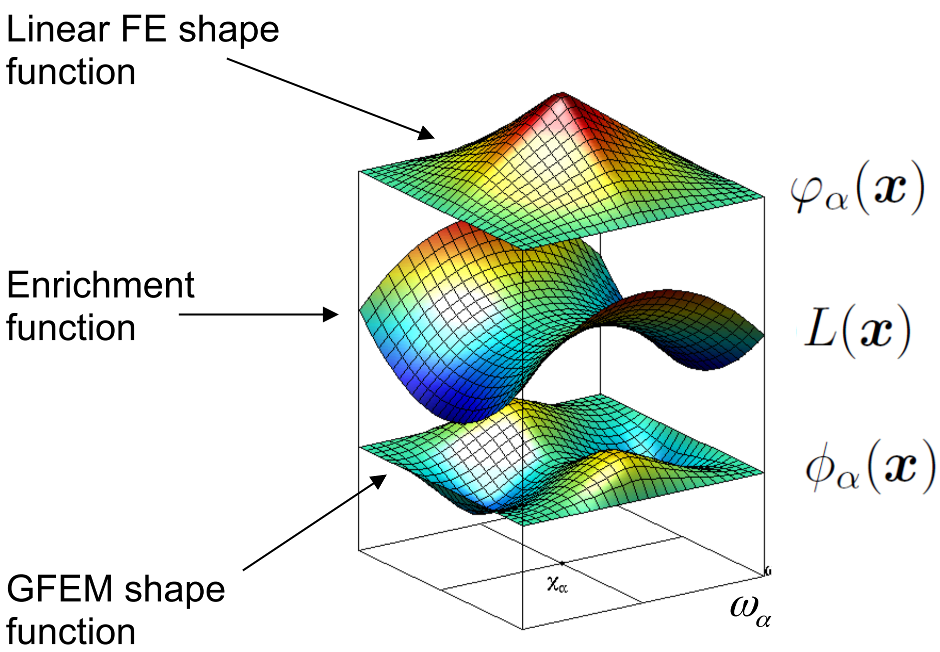
  Belytschko T., Gracie R. and Ventura G. A review of extended/generalized finite element methods for material modeling, *Mod. Simul. Matl. Sci. Eng.,* 2009

  "The XFEM and GFEM are basically identical methods: the name generalized finite element method was adopted by the Texas school in 1995–1996 and the name extended finite element method was coined by the Northwestern school in 1999."

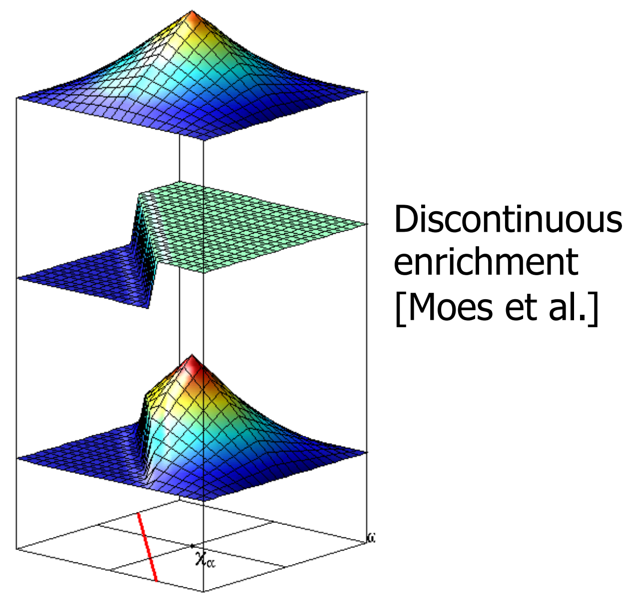
Generalized Finite Element Method

- GFEM shape function = FE shape function * enrichment function

$$\boxed{\phi_\alpha(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x}) L(\boldsymbol{x})}$$

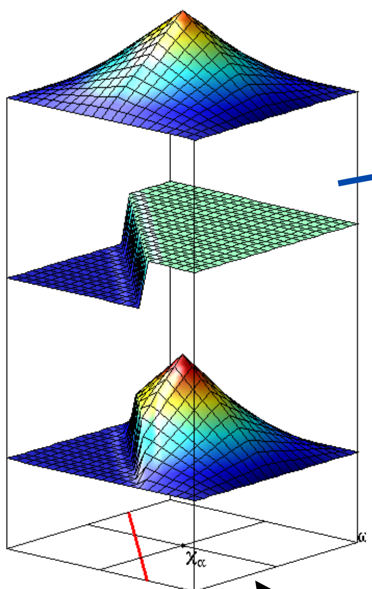


- Allows construction of shape functions incorporating a-priori knowledge about solution



GFEM Approximation for 3-D Cracks

$$\boldsymbol{X}^{hp}(\Omega) = \left\{ \boldsymbol{u} = \sum_{\alpha=1}^{N} \underbrace{\varphi_\alpha(\boldsymbol{x})}_{\text{PoU}} \left[\underbrace{\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{polynomial}} + \underbrace{\mathcal{H}\tilde{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{discontinuous}} + \underbrace{\breve{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{singular}} \right] \right\}$$

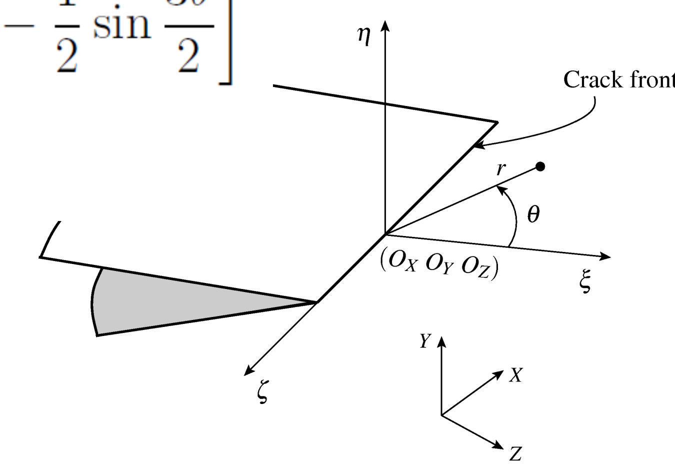


$$\breve{L}^{\xi}_{\alpha 1}(r,\theta) = \sqrt{r}\left[\left(\kappa - \frac{1}{2}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right]$$

$$\breve{L}^{\eta}_{\alpha 1}(r,\theta) = \sqrt{r}\left[\left(\kappa + \frac{1}{2}\right)\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2}\right]$$

$$\breve{L}^{\zeta}_{\alpha 1}(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}$$

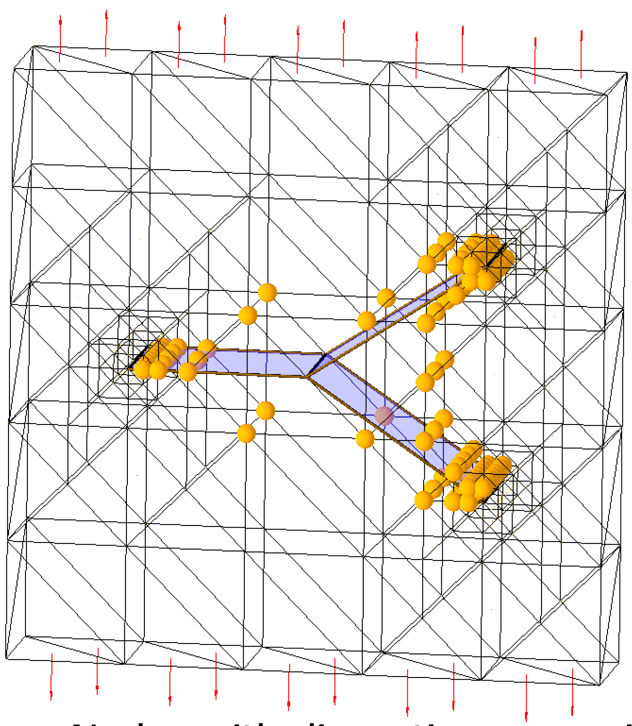
[Duarte and Oden 1996]



cloud or patch 'α

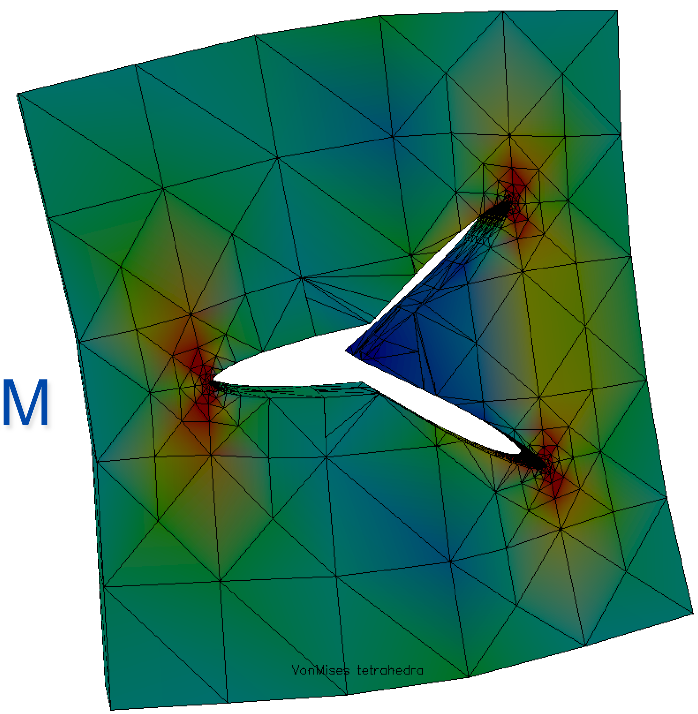
Modeling Cracks with hp-GFEM

- Discontinuities modeled via enrichment functions, *not* the FEM mesh
- Mesh refinement *still required* for acceptable accuracy



● = Nodes with discontinuous enrichments

hp-GFEM

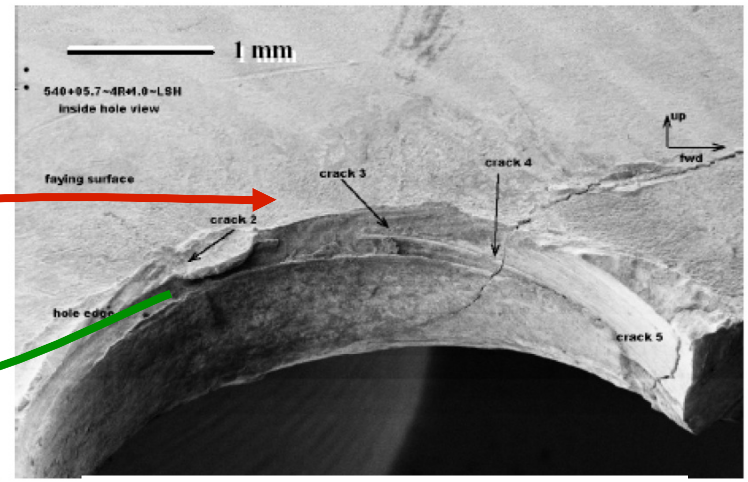


Von Mises stress

[Duarte et al., International Journal Numerical Methods in Engineering, 2007]

Bridging Scales with Global-Local Enrichment Functions

- How to account for interactions among scales?



Multiple cracks around a rivet hole

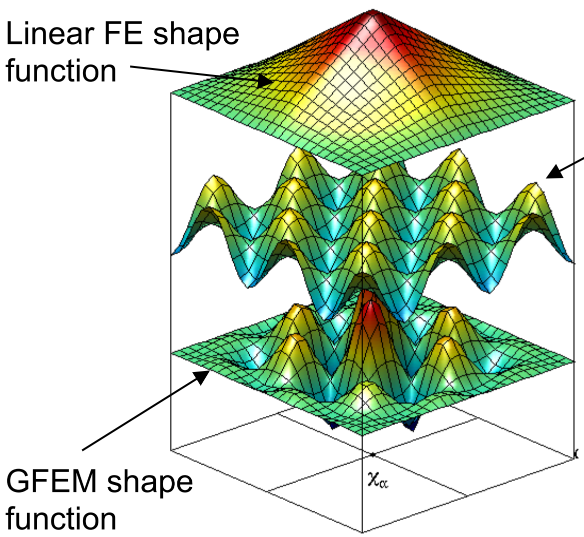
[Sandia National Lab, 2005]

Goal:

- Capture fine scale effects on *coarse* meshes at the global (<u>structural</u>) scale

Bridging Scales with Global-Local Enrichment Functions *

- Enrichment functions computed from solution of local boundary value problems: Global-Local enrichment functions



Enrichment = Numerical solutions of BVP

- **Idea:** *Use available numerical solution at a simulation step to build shape functions for next step* (quasi-static, transient, non-linear, etc.)
- Enrichment functions are produced numerically on-the-fly through a global-local analysis
- Use a *coarse* mesh enriched with Global-Local (G-L) functions

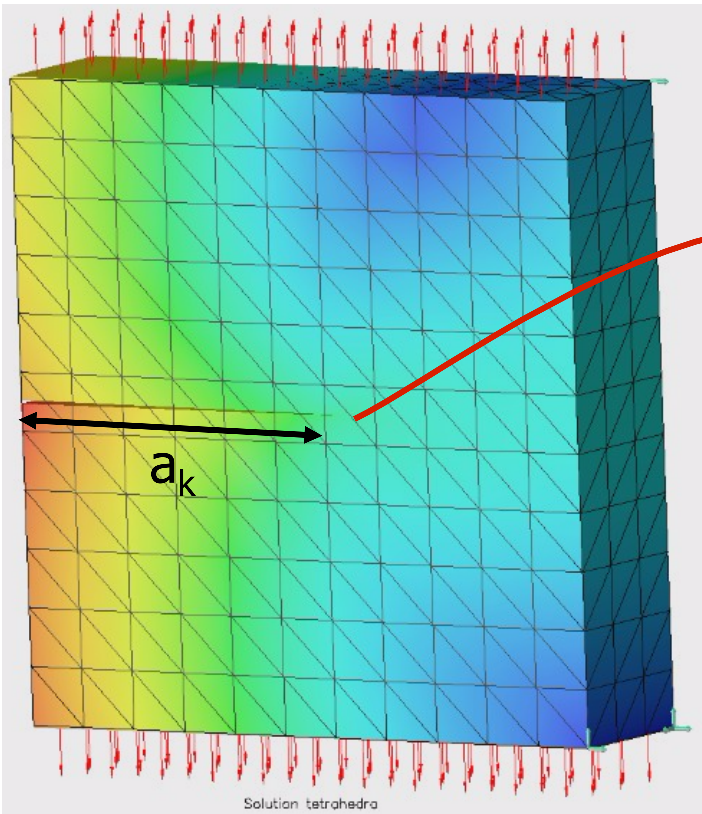
* Duarte et al. 2005, 2007, 2008, 2010, 2011

Related Approaches

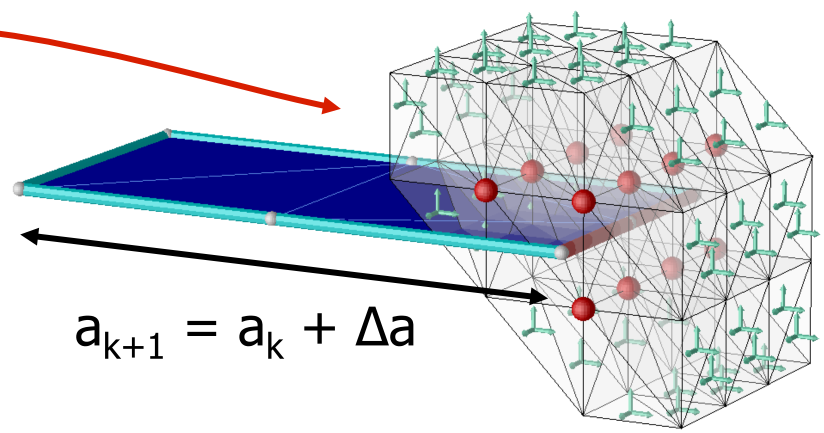
- Global-local FEM developed in the 1970's
- Multiscale FEM of Hou and Wu, 1997
- Mesh-based handbook approach of Strouboulis et al., 2001
- Multiscale method of Krause and Rank, 2003
- Two-scale XFEM for 2-D cracks, Cloirec et al., 2005
- Multiscale projection method, Loehnert and Belytschko, 2007
- Multiscale XFEM crack propagation, Guidault et al., 2008
- Spider-XFEM, Chahine et al., 2008
- Reduced basis enrichment for the XFEM, Chahine et al., 2008
- Local multigrid X-FEM for 3-D cracks, Rannou et al., 2009
- Method of Menk and Bordas for fracture of bi-materials, 2010
- Harmonic enrichments for 2-D branched cracks, Mousavi et al., 2011

Global-Local Enrichments for 3-D Fractures

- u_G^k solution of global problem at crack step k



- Define local domain containing crack front at step k+1

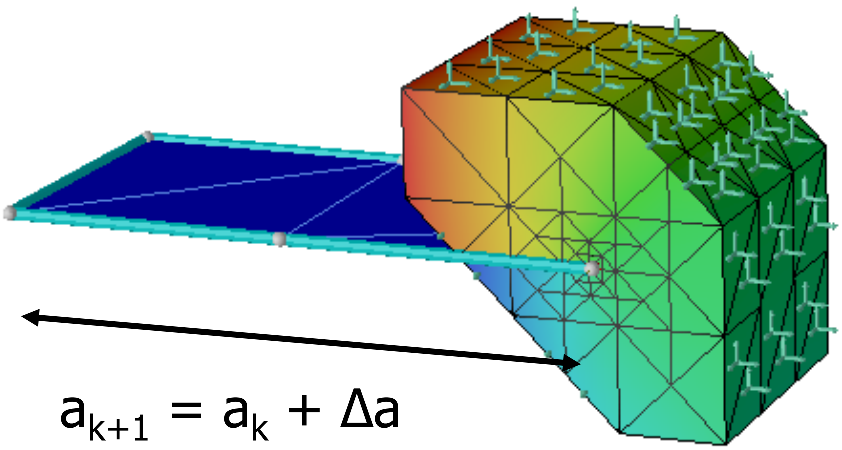


Local problem with crack size a_{k+1}

$u_G^k \in X_G^k(\Omega)$ = solution of global problem with crack size a_k

Global-Local Enrichments for 3-D Fractures

- Solve local problem at step k using *hp*-GFEM



$a_{k+1} = a_k + \Delta a$

Boundary conditions for local problems provided by global solution:

$$\boldsymbol{u}_L^k = \boldsymbol{u}_G^k \quad \text{on} \quad \partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)$$

$$\boldsymbol{X}_L^k\left(\Omega_L^k\right) = hp\text{-GFEM space}$$

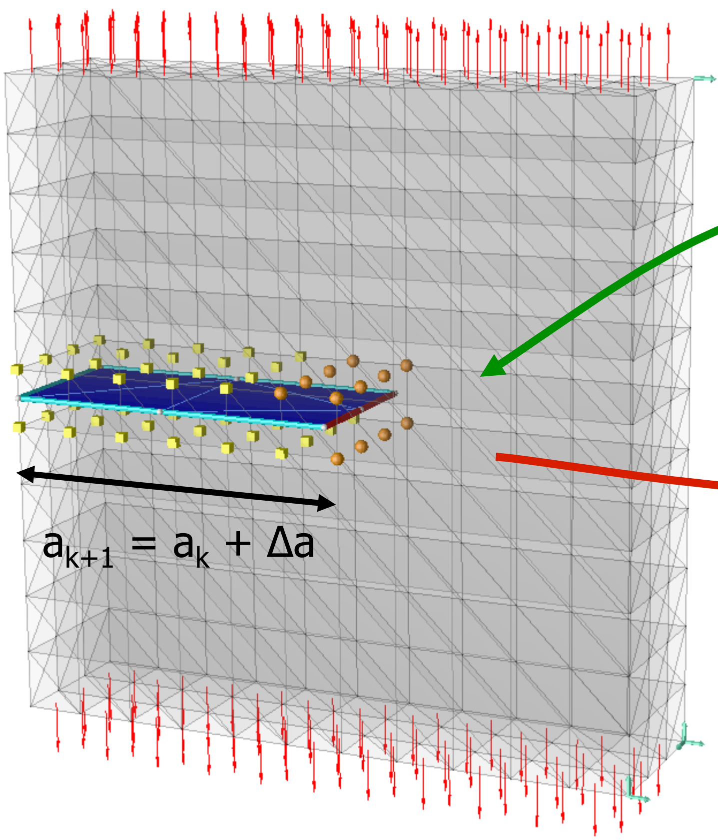
Find $\boldsymbol{u}_L^k \in \boldsymbol{X}_L^k\left(\Omega_L^k\right) \subset H^1\left(\Omega_L^k\right)$ such that $\forall\, \boldsymbol{v}_L^k \in \boldsymbol{X}_L^k\left(\Omega_L^k\right)$

$$\int_{\Omega_L^k} \boldsymbol{\sigma}(\boldsymbol{u}_L^k) : \boldsymbol{\varepsilon}(\boldsymbol{v}_L^k) d\boldsymbol{x} + \kappa \int_{\partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)} \boldsymbol{u}_L^k \cdot \boldsymbol{v}_L^k ds$$

$$= \int_{\partial\Omega_L^k \cap \partial\Omega^\sigma} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}_L^k ds + \kappa \int_{\partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)} \boldsymbol{u}_G^k \cdot \boldsymbol{v}_L^k ds$$

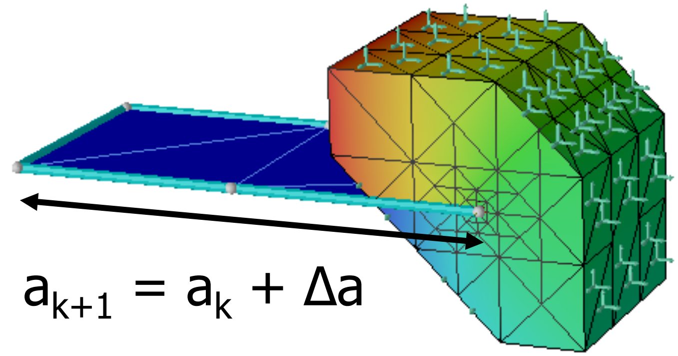
Global-Local Enrichments for 3-D Fractures

- **Defining Step:** Global space is enriched with local solutions

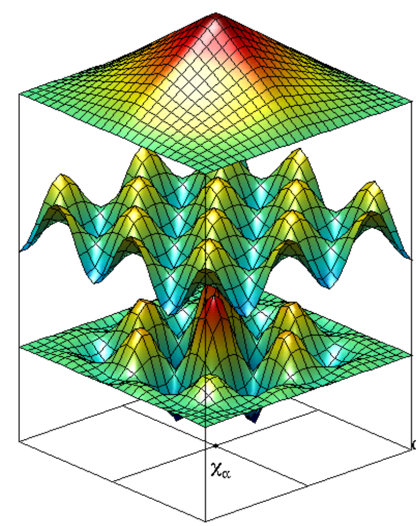


global-local enrichments

BC for local problem



$$\phi_{\alpha}^{k+1}(\boldsymbol{x}) = \varphi_{\alpha}(\boldsymbol{x})\boldsymbol{u}_L^k(\boldsymbol{x})$$

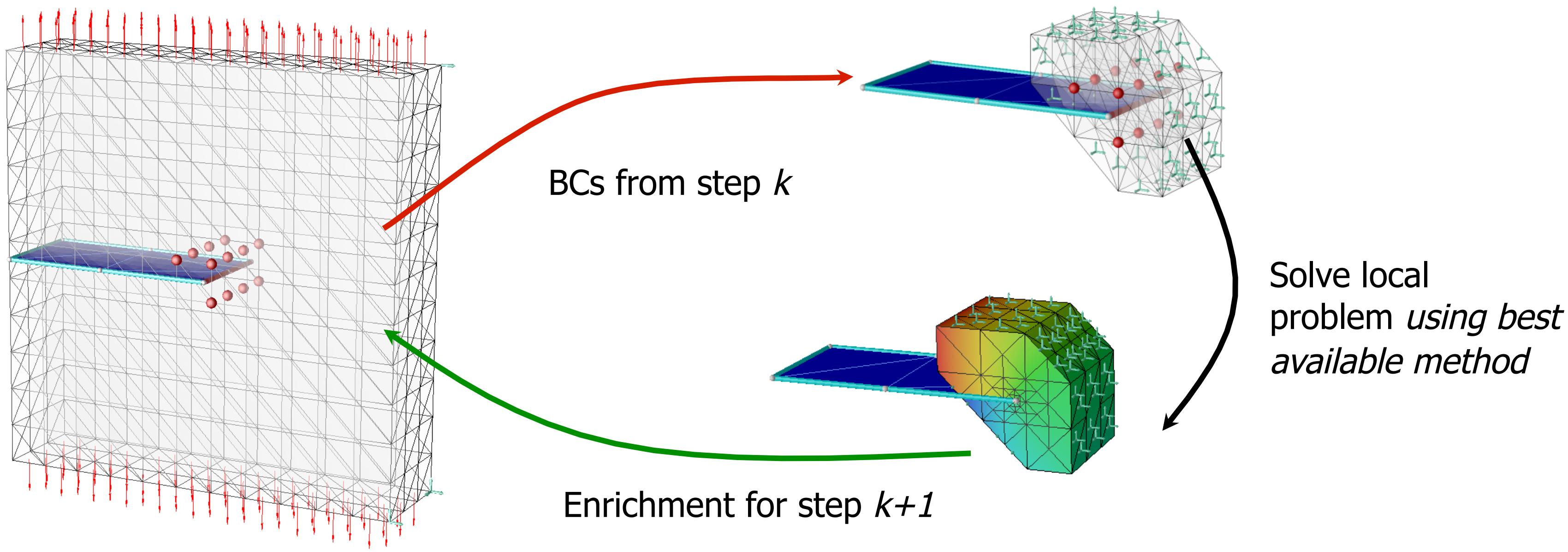


$\boldsymbol{u}_G^{k+1} \in \boldsymbol{X}_G^{k+1}(\Omega)$ = solution of global problem with crack size a_{k+1}

- Procedure may be repeated: Update local BCs and enrichment functions

Global-Local Enrichments for 3-D Fractures

- **Summary:** Use solution of global problem at simulation *k* to build enrichment functions for step *k+1*



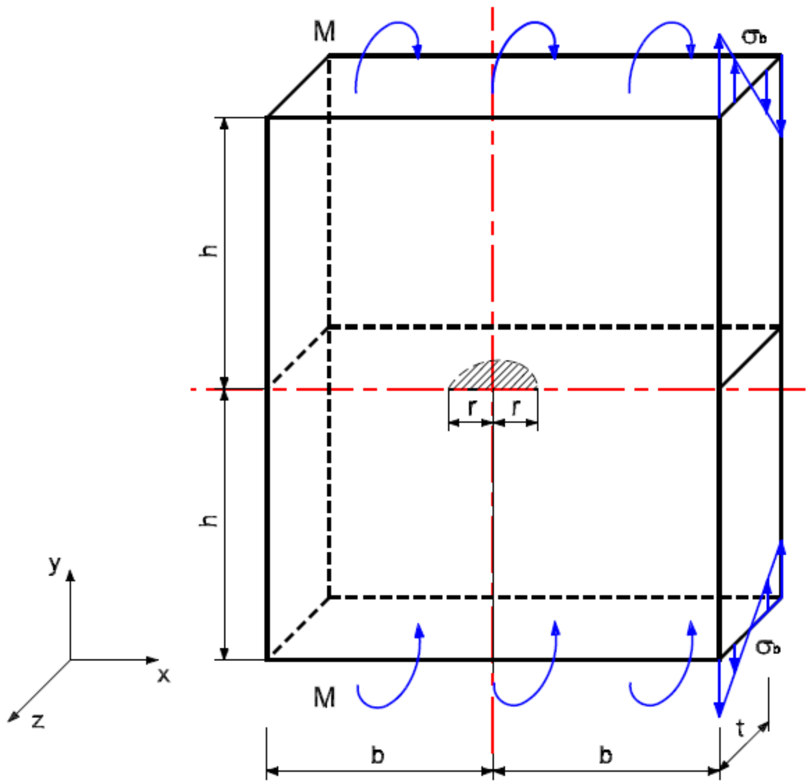
• Discretization spaces updated on-the-fly with global-local enrichment functions

$$\boldsymbol{X}_G^{k+1}(\Omega_G) = \left\{ \boldsymbol{u} = \underbrace{\sum_{\alpha=1}^{N} \varphi_\alpha(\boldsymbol{x})\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{coarse-scale approx.}} + \underbrace{\sum_{\beta \in \mathcal{I}_{gl}^k} \varphi_\beta(\boldsymbol{x})\boldsymbol{u}_\beta^{gl(k)}(\boldsymbol{x})}_{\text{fine-scale approx.}} \right\}$$

$\boldsymbol{u}_\beta^{gl(k)}$ = G-L enrichment

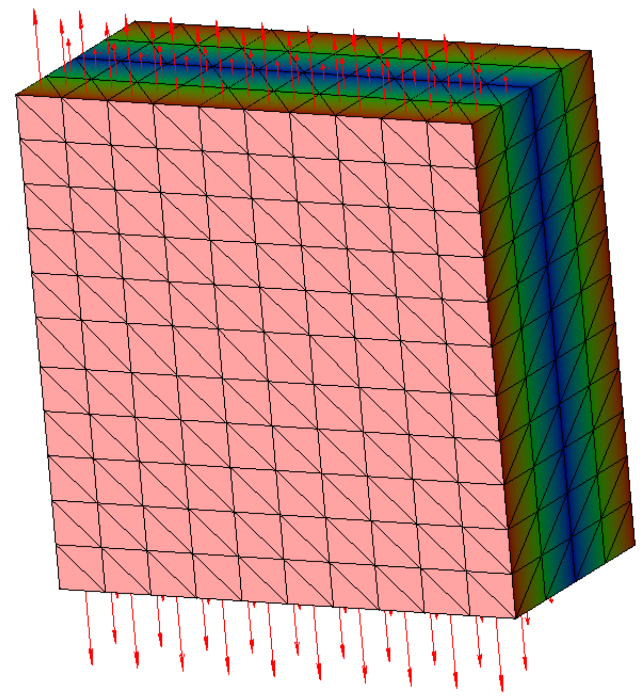
Numerical Verification: Static Crack

- Plate with a surface crack



b/t = h/t = 1, r/b=0.2

- **Initial (un-cracked) global problem**



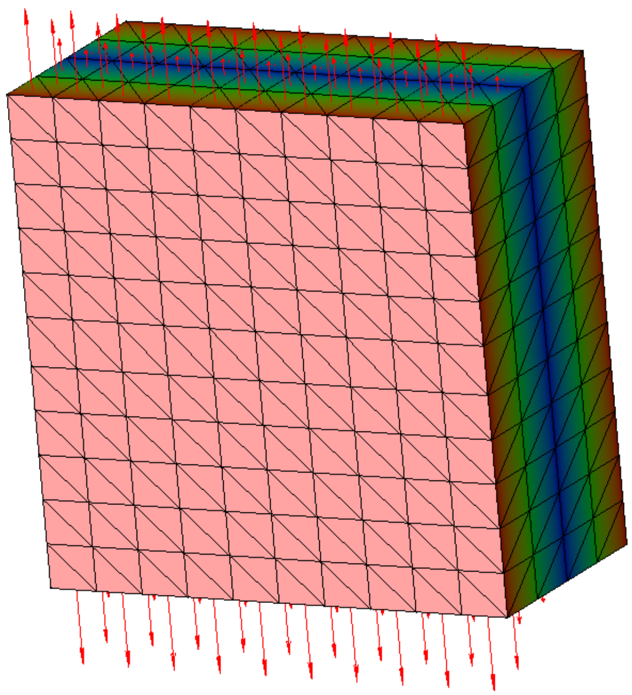
Solution of un-cracked global problem

- Crack *not* modeled in initial global problem
- Goal: Analyze cracked domain while keeping global model as it is

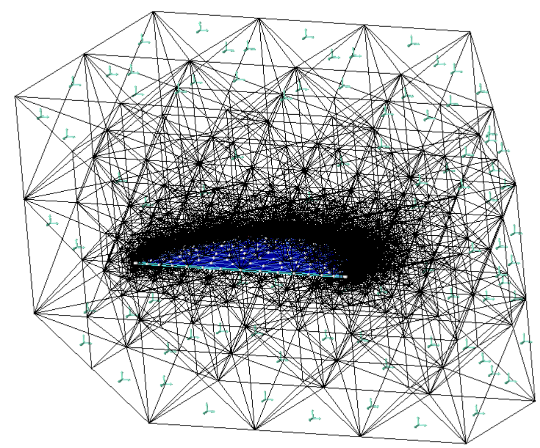
GFEM with G-L Enrichment Functions

- **Local problem**: Define and solve with the GFEM a local problem containing crack

Single local problem for entire crack



Boundary conditions



Boundary conditions for local problem provided by global solution:

$$\boxed{\boldsymbol{u}_{loc} = \boldsymbol{u}_G^0 \quad \text{on } \partial\Omega_{loc}}$$

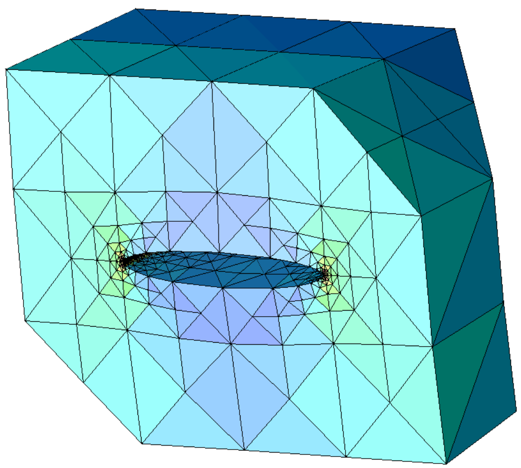
$\boldsymbol{u}_G^0$ = solution of global problem

Other types of BCs can be used. E.g. Spring BC:

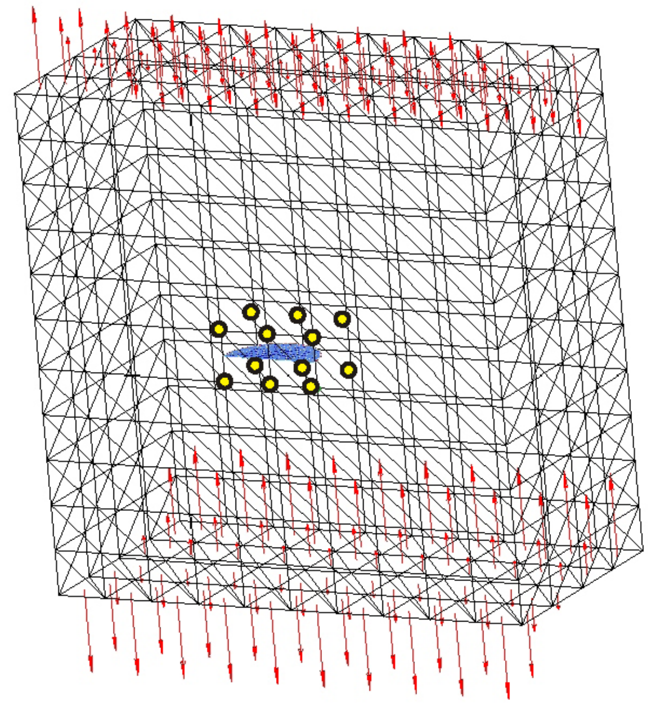
$$\boldsymbol{t}(\boldsymbol{u}) = \boldsymbol{t}(\boldsymbol{u}_G^0) + \kappa \boldsymbol{u}_G^0 - \kappa \boldsymbol{u}$$

GFEM with G-L Enrichment Functions

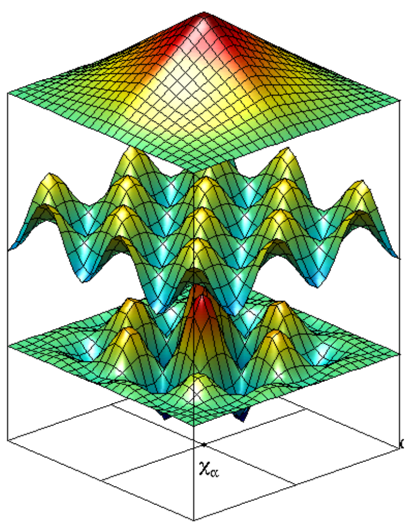
- **Enriched global problem**



Global-local enrichments



Enriched global problem



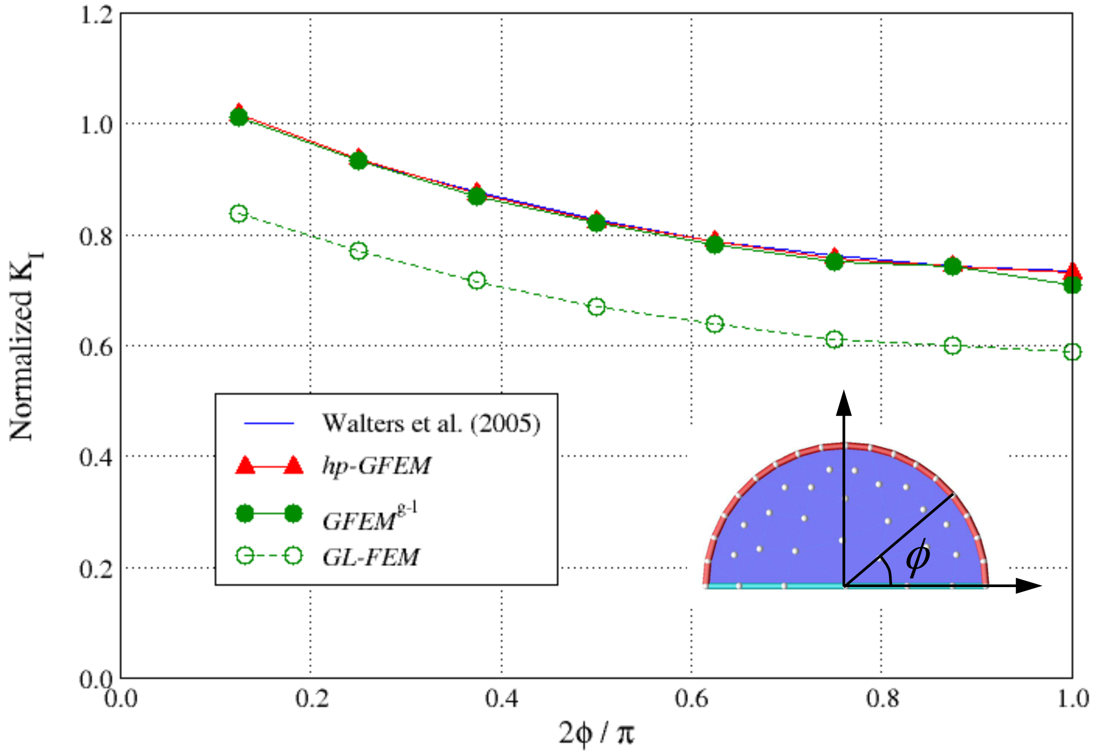
Enrichment of global FEM mesh with local solutions

$$\boxed{\phi_\alpha = \varphi_\alpha \boldsymbol{u}_{loc}}$$

- Only 36 dofs added

 = 0.2 % (out of 19,800)

Numerical Verification: Static Crack

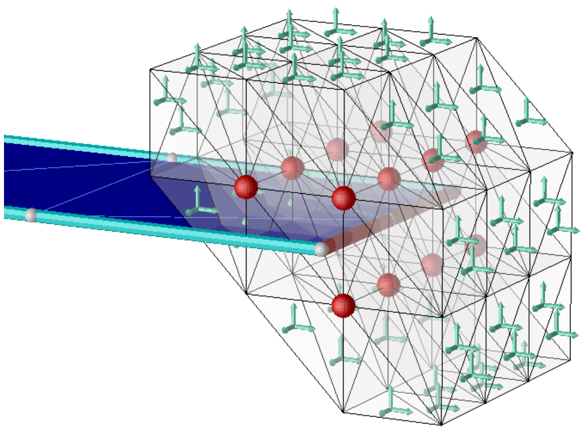


Normalized K_I for *GL-FEM* and *GFEM* $^{g\text{-}l}$ w/ displacement BC in local problem♪

$$e^r(K_i) := \frac{\|e_i\|_{L^2}}{\|\hat{K}_i\|_{L^2}} = \frac{\sqrt{\sum_{j=1}^{N_{\text{ext}}} \left(K_i^j - \hat{K}_i^j\right)^2}}{\sqrt{\sum_{j=1}^{N_{\text{ext}}} \left(\hat{K}_i^j\right)^2}}$$

- Relative error K_I *GFEM* $^{g\text{-}l}$ ≈ 1.2 %
- Relative error K_I *GL-FEM* ≈ 18.5 %

Local Problem and Spring BC



$\boldsymbol{t}(\boldsymbol{u}) = \kappa(\boldsymbol{\delta} - \boldsymbol{u})$ [Szabo and Babuska, 1991]

$\kappa\boldsymbol{\delta} := \boldsymbol{t}(\boldsymbol{u}_G^0) + \kappa\boldsymbol{u}_G^0$

$\boldsymbol{\delta}$: Displacement imposed at base of spring system

Spring BC: $\boldsymbol{t}(\boldsymbol{u}) = \boldsymbol{t}(\boldsymbol{u}_G^0) + \kappa\boldsymbol{u}_G^0 - \kappa\boldsymbol{u}$

$$\boldsymbol{t}(\boldsymbol{u}_G^0) = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_G^0) = \hat{\boldsymbol{n}} \cdot (\boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}_G^0))$$

(i) Neumann boundary condition: Set $\kappa = 0$.

(ii) Dirichlet boundary condition: Set $\kappa = \eta \gg 1$.

(iii) Cauchy or spring boundary condition: Set $0 < \kappa < \eta$.

$$\int_{\Omega_L} \boldsymbol{\sigma}(\boldsymbol{u}_L) : \boldsymbol{\varepsilon}(\boldsymbol{v}_L) d\boldsymbol{x} + \eta \int_{\partial\Omega_L \cap \partial\Omega_G^u} \boldsymbol{u}_L \cdot \boldsymbol{v}_L d\boldsymbol{s} + \kappa \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} \boldsymbol{u}_L \cdot \boldsymbol{v}_L d\boldsymbol{s} =$$

$$\int_{\partial\Omega_L \cap \partial\Omega_G^\sigma} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}_L d\boldsymbol{s} + \eta \int_{\partial\Omega_L \cap \partial\Omega_G^u} \bar{\boldsymbol{u}} \cdot \boldsymbol{v}_L d\boldsymbol{s} + \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} (\boldsymbol{t}(\boldsymbol{u}_G^0) + \kappa\boldsymbol{u}_G^0) \cdot \boldsymbol{v}_L d\boldsymbol{s}$$

$\boldsymbol{u}_L$: Local solution

$\boldsymbol{u}_G^0$: Solution of initial global problem

Local Problem and Spring BC

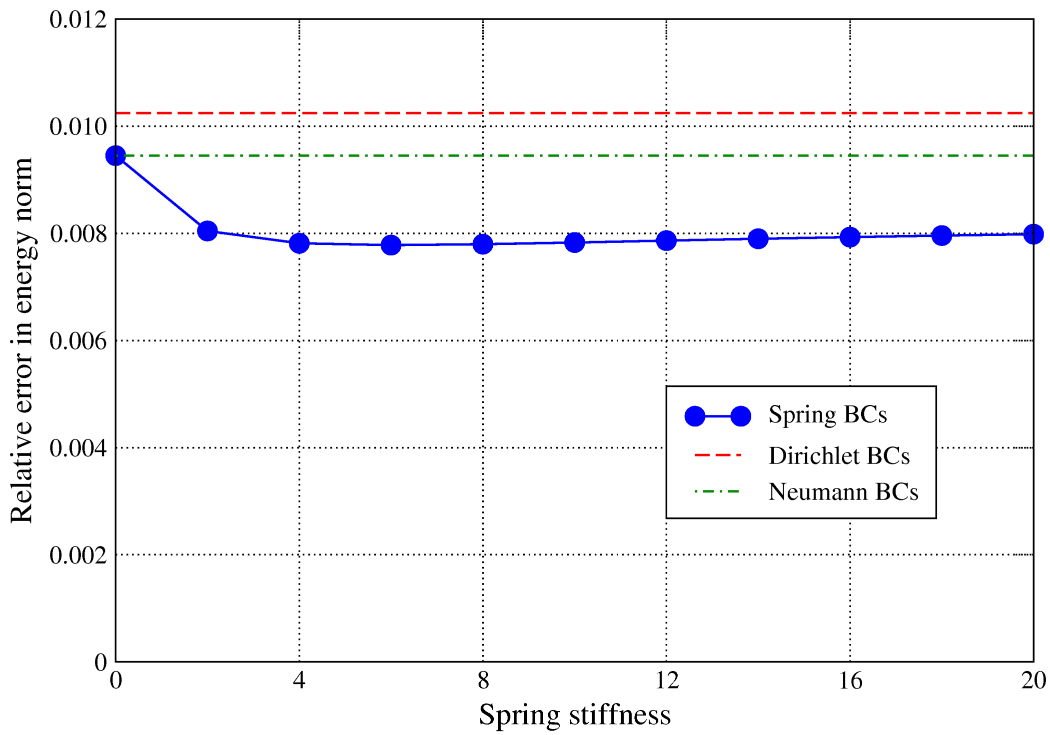
- **Sensitivity analysis to stiffness of spring boundary condition**

Spring stiffness

1) Neumann BC: $\kappa = 0$
2) Spring BC: $0 < \kappa < \eta$
3) Dirichlet BC: $\kappa = \eta \gg 1$

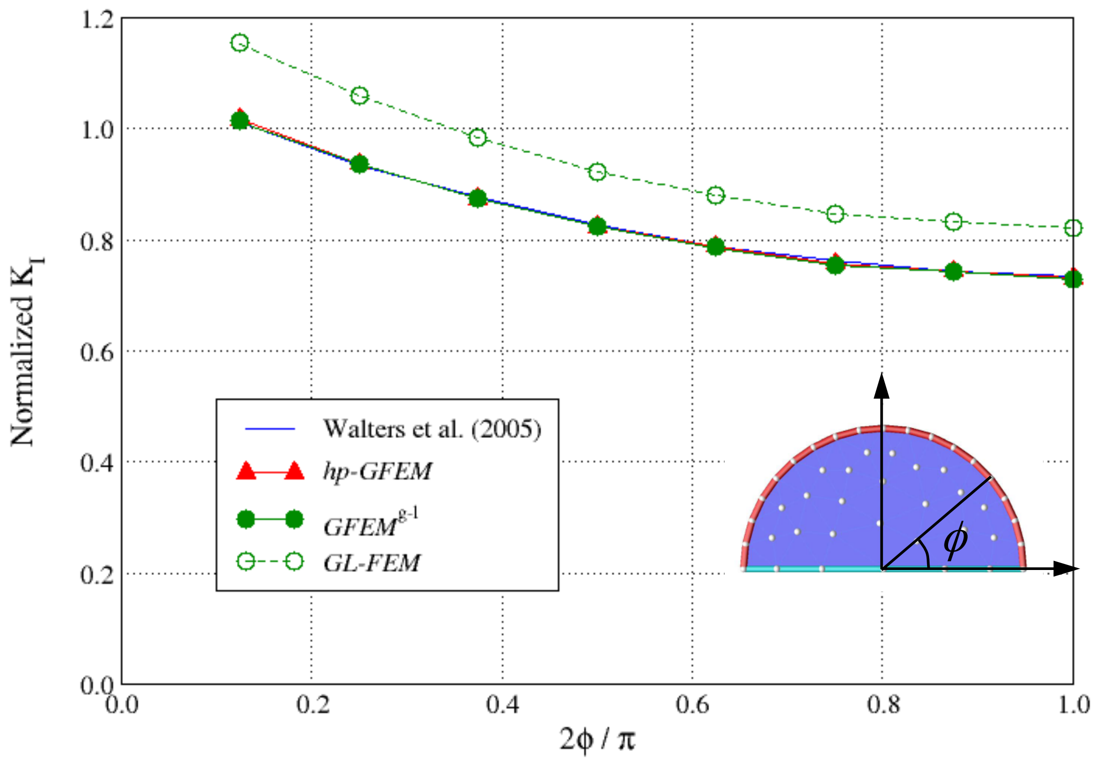
Spring stiffness for spring BC

$$\kappa = \frac{E}{\sqrt[n]{V_0 J}} = 8.7358$$



- Spring boundary conditions provide more accurate solutions
- Low sensitivity to spring constant: Robustness of method

Local Problem and Spring BC

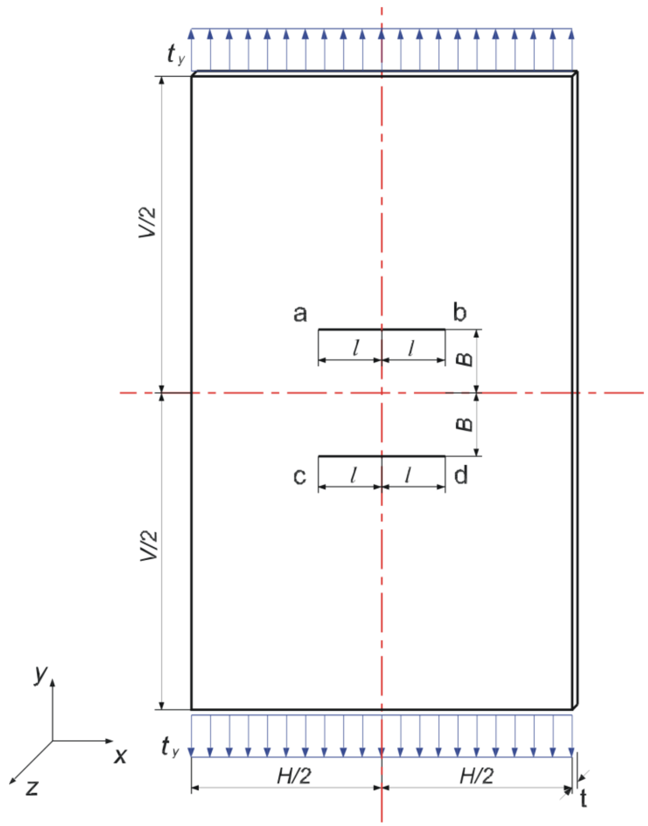


Normalized K_I for *GL-FEM* and *GFEM* $^{g\text{-}l}$ w/ spring BC in local problem

- Relative error K_I *hp-GFEM* ≈ 0.4 %
- Relative error K_I *GFEM* $^{g\text{-}l}$ ≈ 0.5 %
- Relative error K_I *GL-FEM* ≈ 12.4 %

Robustness: Comparison with Global-Local Finite Element Method

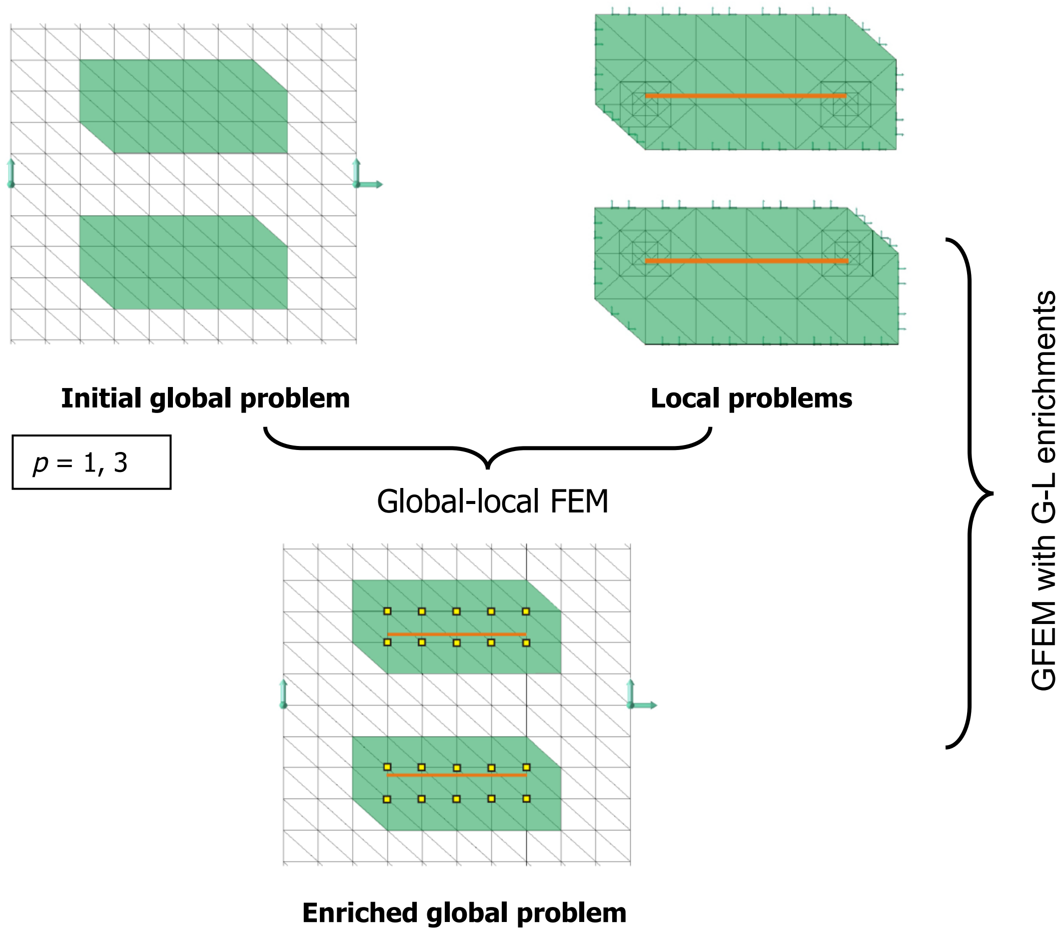
- Interacting cracks



Analysis for varying B/H

$2l$ = 4.0; V = 200.0; H = 10.0; t = 1.0
Poisson's ratio = 0.0
Young's modulus = 200,000

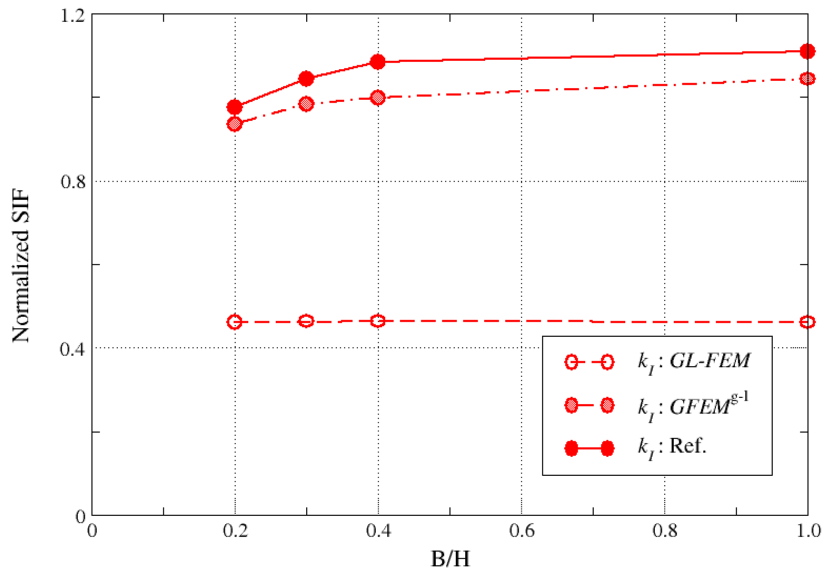
Two interacting cracks



Cracks NOT discretized in global domain

Comparison with GL-FEM

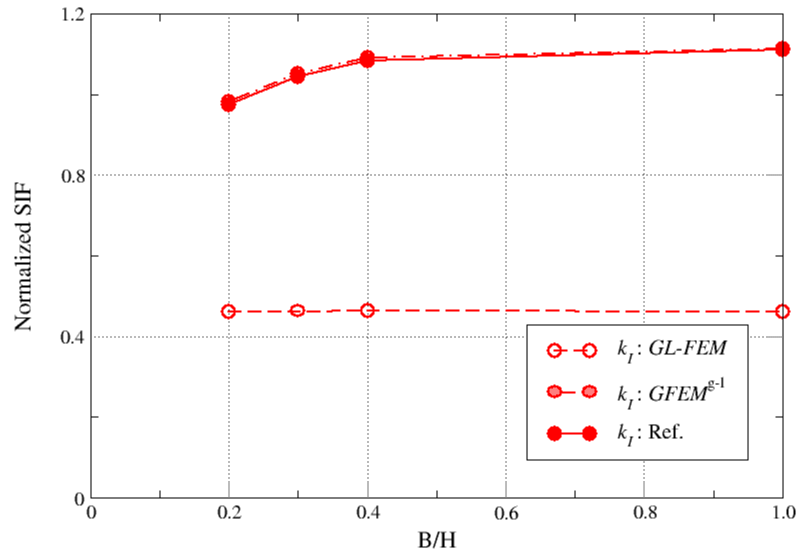
- Normalized Mode I SIF. p = 1 in the global problem



B/H	Mode I, p = 1 in the global problem			
	SIF (GL-FEM)	Rel. err(%)	SIF (GFEM)	Rel. err(%)
0.2	0.4617	52.64	0.9354	4.05
0.3	0.4625	55.69	0.9834	5.78
0.4	0.4630	57.28	0.9987	7.86
1.0	0.4604	58.51	1.0425	6.05

Comparison with GL-FEM

- Improving BCs for local problems: p = 3 in global problem



B/H	Mode I, p = 3 in global problem			
	SIF (GL-FEM)	Rel. err(%)	SIF (GFEM)	Rel. err(%)
0.2	0.4617	52.64	0.9807	-0.59
0.3	0.4625	55.69	1.0517	-0.77
0.4	0.4630	57.28	1.0902	-0.58
1.0	0.4604	58.51	1.1125	-0.26

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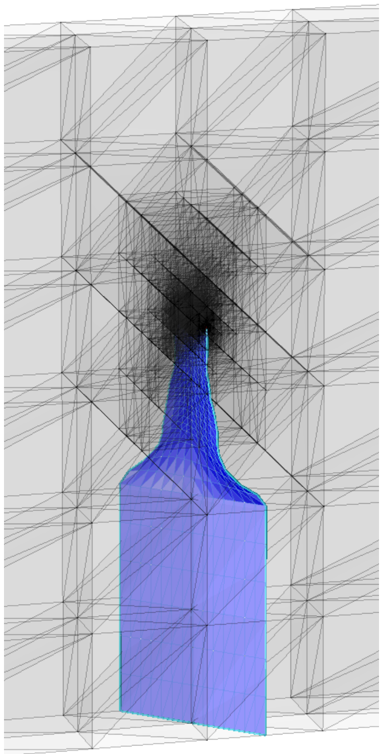
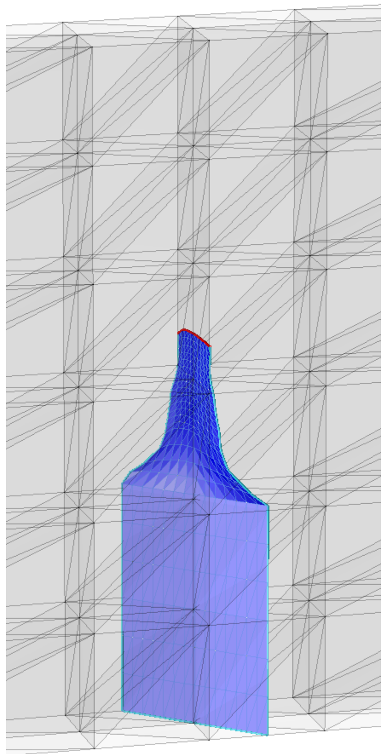
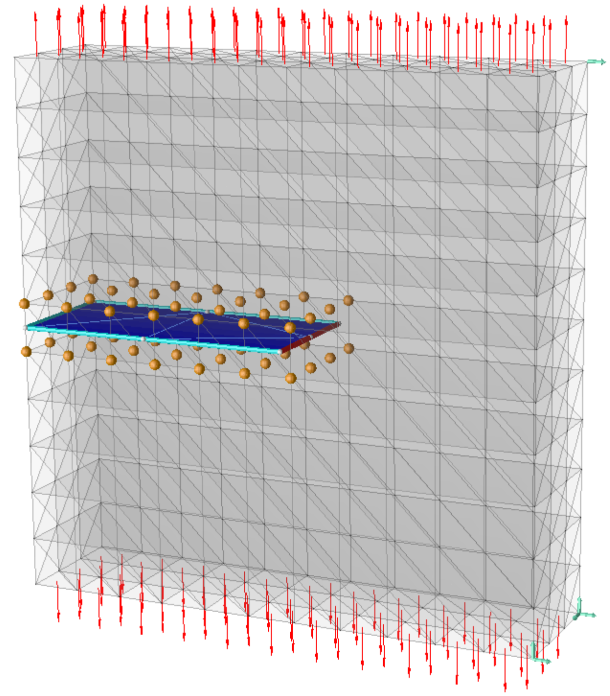
Mathematical Analysis*



hp-GFEM/FEM



GFEM^{gl}



GFEM^{gl}: Error controlled through global-local enrichments

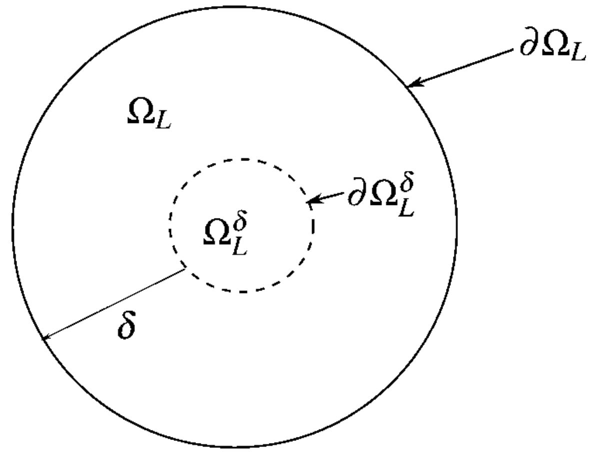
Questions:

- What are the effects of inexact BCs at fine-scale problems?
- How to control them?

*with V. Gupta

A-Priori Error Estimate

- Local error estimate



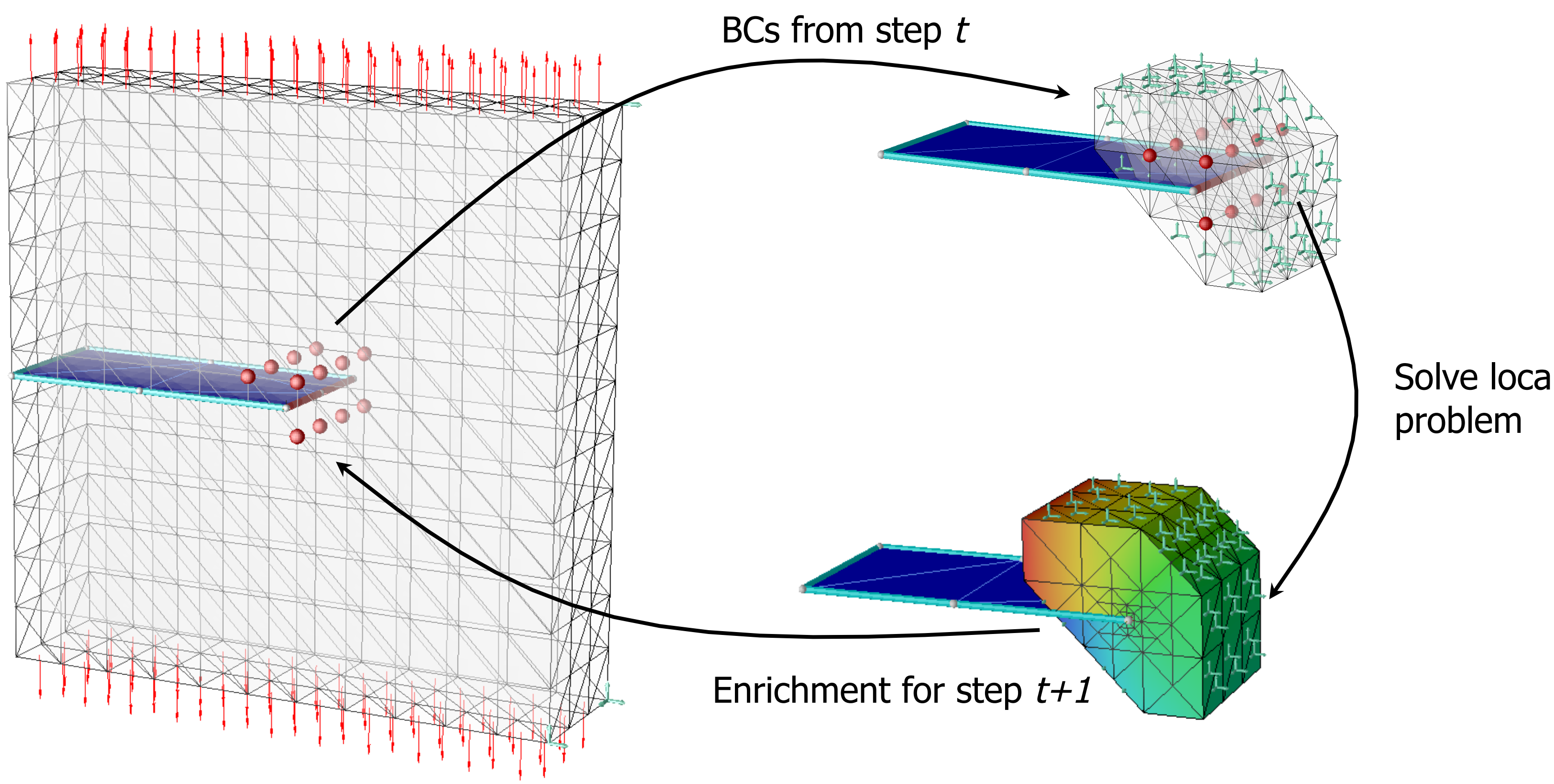
$$\underbrace{\|\boldsymbol{u}^{exBC} - \boldsymbol{u}_h^{inexBC}\|_{\varepsilon(\Omega_L^\delta)} \leq C \inf_{\boldsymbol{x}\in \boldsymbol{X}_L^{hp}(\Omega_L)} \|\boldsymbol{u}^{inexBC} - \boldsymbol{x}\|_{\varepsilon(\Omega_L)}}_{\text{Discretization error}} + \underbrace{\frac{C_1}{\delta}\|\boldsymbol{u}^{exBC} - \boldsymbol{u}^{inexBC}\|_{L^2(\Omega_L)}}_{\text{Effect of inexact BC}}$$

- Global Error [Babuska and Melenk, 1996]

$$\|\boldsymbol{u}-\boldsymbol{u}_G\|^2_{\varepsilon(\Omega)} \leq C\sum_{\alpha=1}^{N} \inf_{\boldsymbol{u}_\alpha\in\chi_\alpha} \|\boldsymbol{u}-\boldsymbol{u}_\alpha\|^2_{\varepsilon(\omega_\alpha)} \leq C\sum_{\alpha=1}^{N} \|\boldsymbol{u}-\boldsymbol{u}_h^{\text{inexBC}}\|^2_{\varepsilon(\omega_\alpha)}$$

where $\boldsymbol{u} \equiv \boldsymbol{u}^{\text{exBC}}$

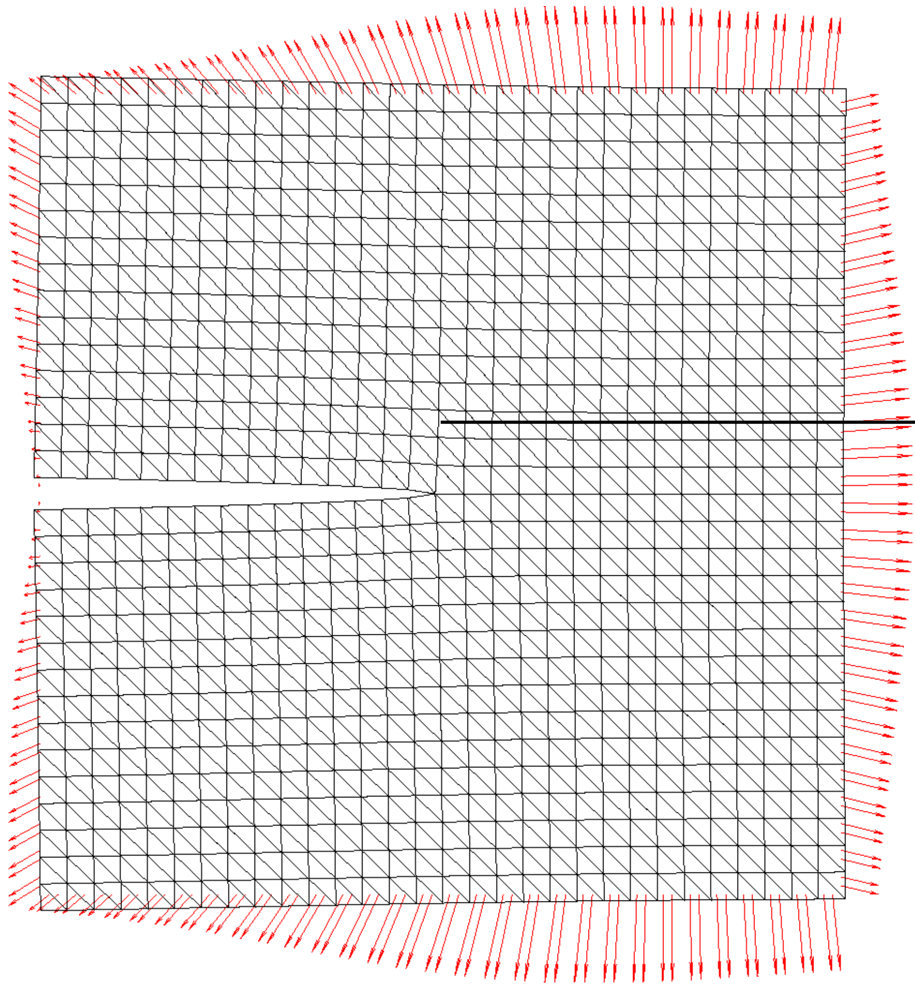
Strategy I: Multiple Global-Local Iterations

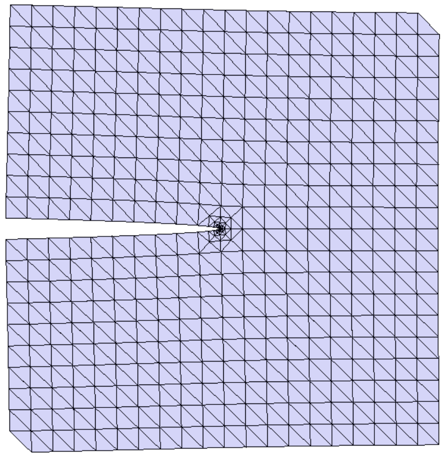


- Repeat Global-local-Global cycle

Strategy I: Multiple Global-Local Iterations

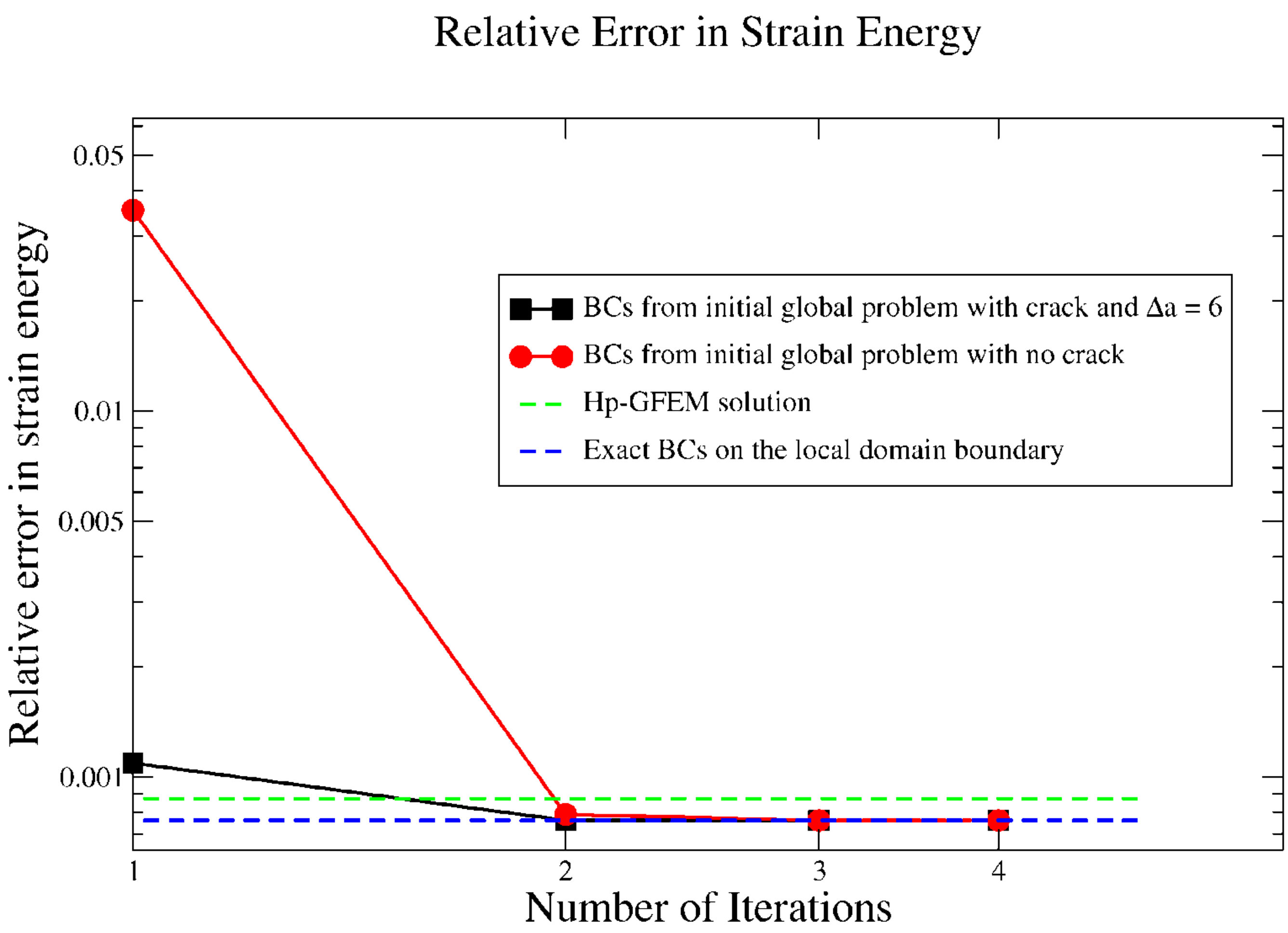
- 30” x 30” x 1” edge-crack panel loaded with Mode I tractions





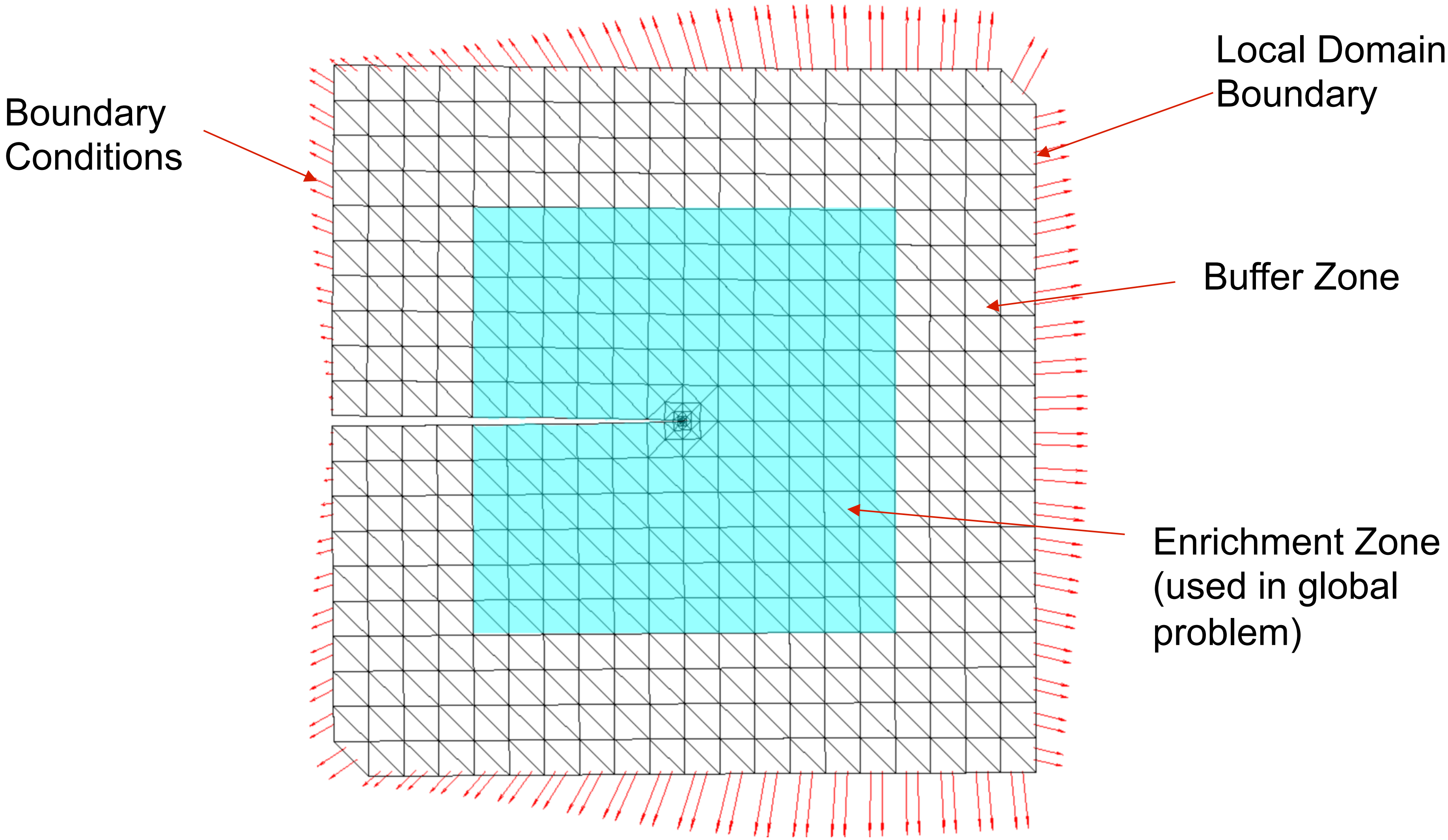
Local Mesh

Strategy I: Multiple Global-Local Iterations



- GFEM^{gl} can deliver same accuracy as hp-GFEM (DNS)

Strategy II: Buffer Zone in Local Domain



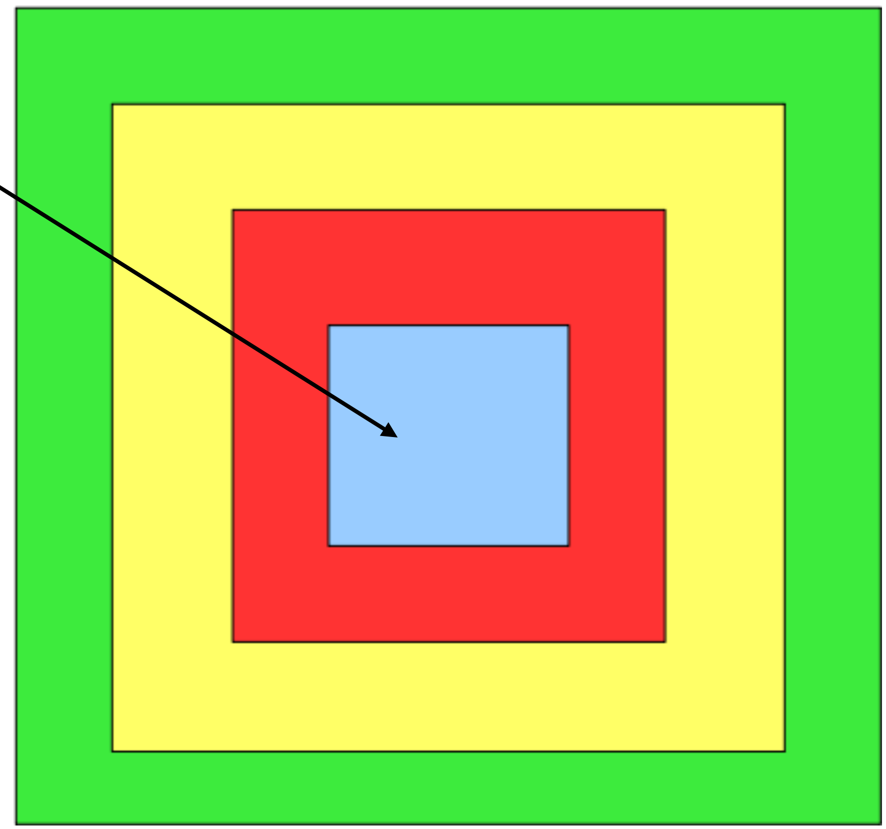
Strategy II: Buffer Zone in Local Domain

- **Buffer Zone Sizes Considered**

  - Enrichment Zone: 4” X 4” blue square region

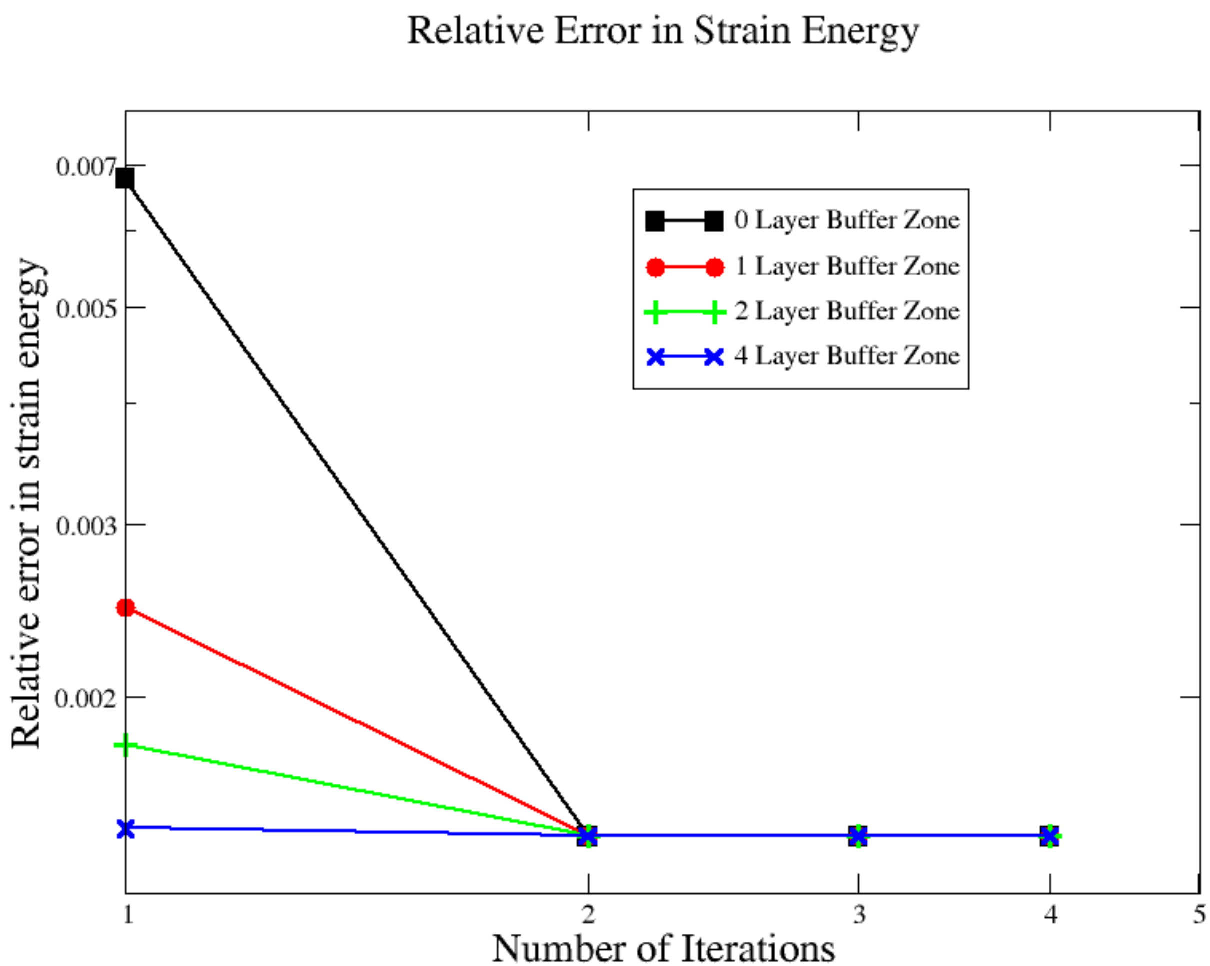
  - Buffer zone (in terms of number of layers of elements):

    - Red - 1 layer
    - Yellow - 2 layers
    - Green - 4 layers



Not to scale

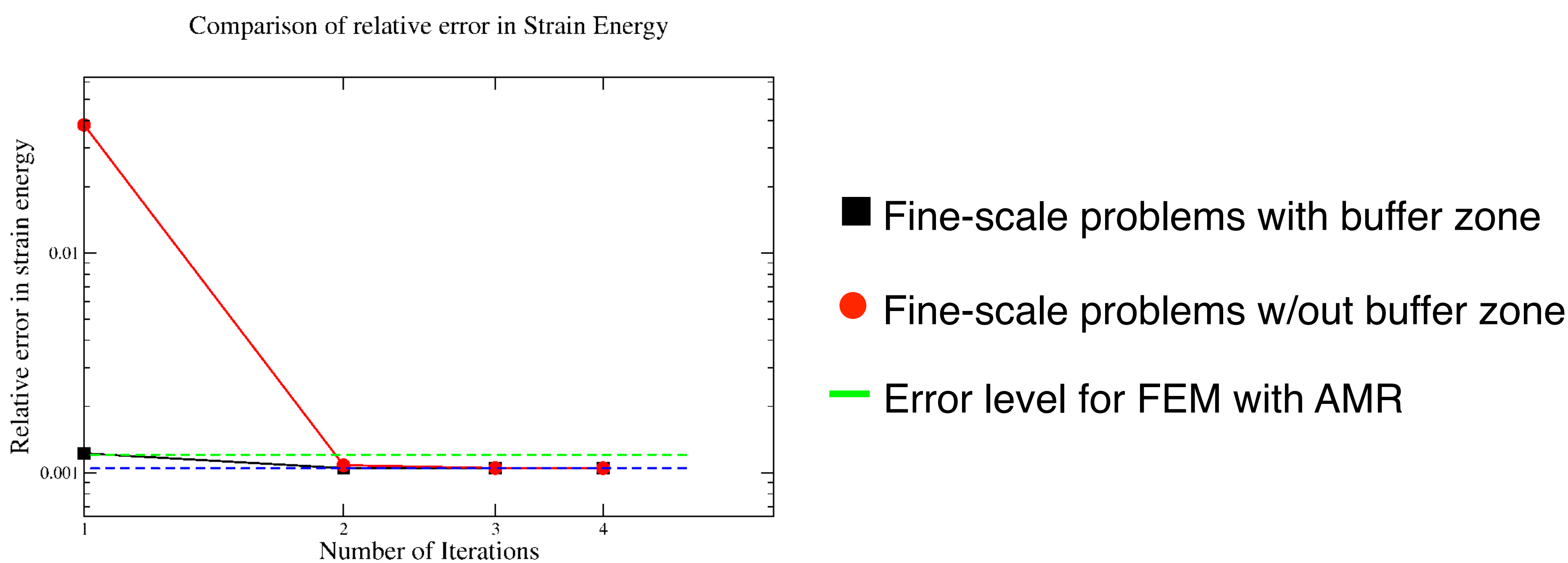
Strategy II: Buffer Zone in Local Domain



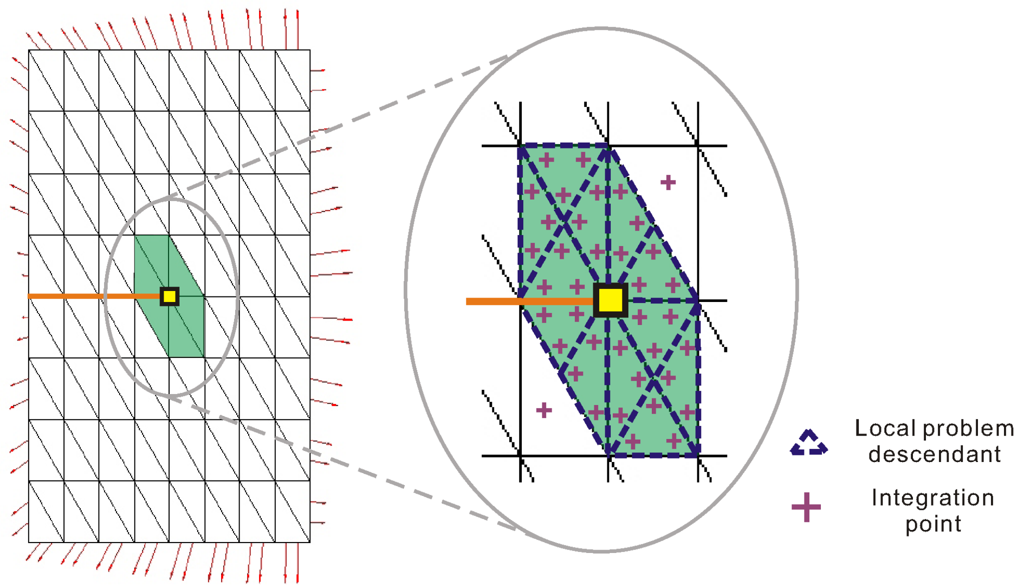
- BCs from global problem *without* a crack

Mathematical Analysis: Summary

- A-priori error estimates and convergence analyses show optimal convergence even on <u>tough</u> problems:
    - Problems with strong singularities and/or numerical pollution effects
- Quality of global-local enrichments can be controlled through
    - Global-local iteration cycles
    - Buffer-zone in local domains



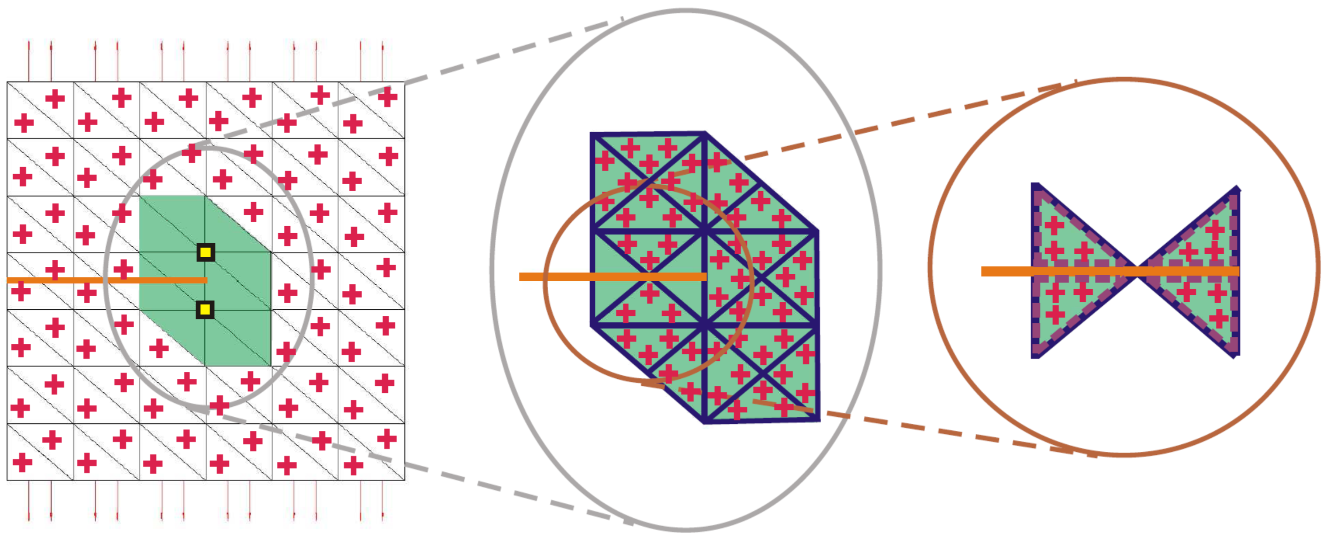
Numerical Integration Procedure



Numerical integration scheme in the global elements enriched with local solution

- Use local mesh for integration.
- Local mesh nested in global mesh: Greatly facilitate implementation.

Numerical Integration Procedure



(a) Global Computational Elements

(b) Local Computational Elements

(c) Local Integration Elements

Numerical integration scheme in the global elements enriched with local solution

- ✓ Use local mesh for integration.
- ✓ Local mesh nested in global mesh: Greatly facilitate implementation.

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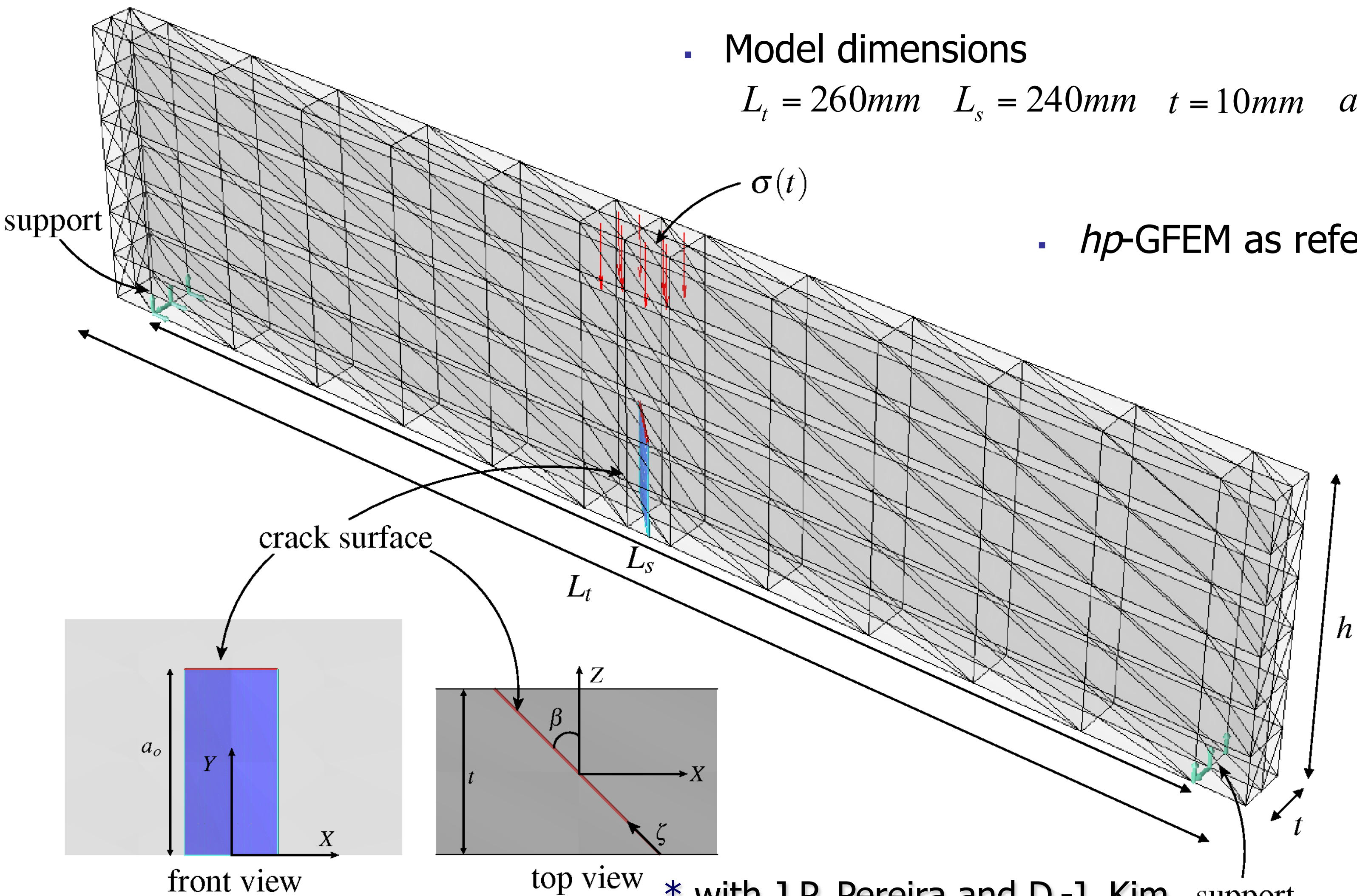
Crack Propagation: Edge-Notched Beam with Slanted Crack *

- Fatigue Crack Growth: hp-GFEM and GFEM^{gl} solutions

  - Model dimensions

    $L_t = 260mm \quad L_s = 240mm \quad t = 10mm \quad a_o/h = 1/3 \quad \beta = 45^o$

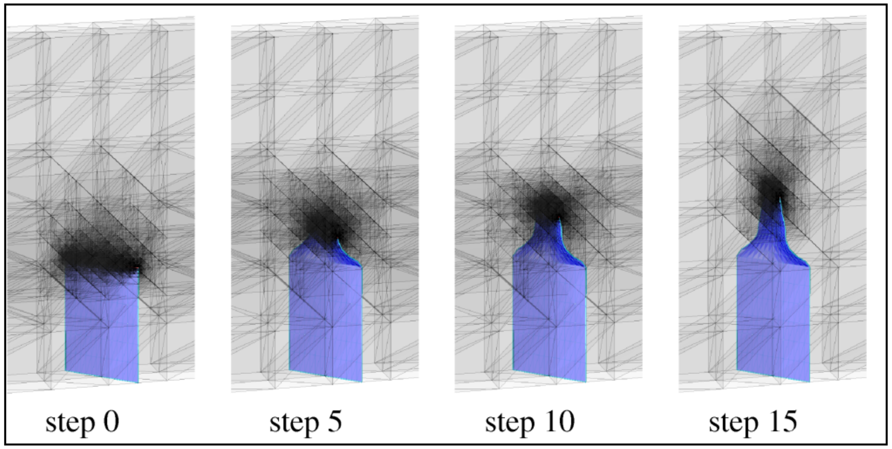
  - *hp*-GFEM as reference solution



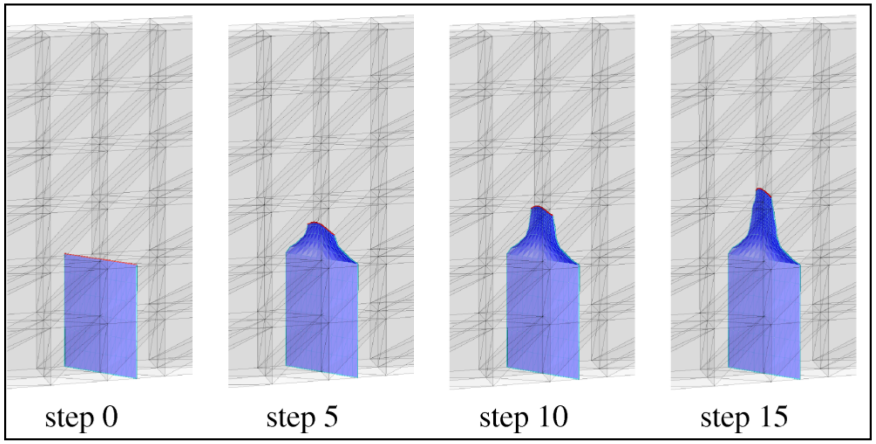
Movie

* with J.P. Pereira and D.-J. Kim

Edge-Notched Beam with Slanted Crack



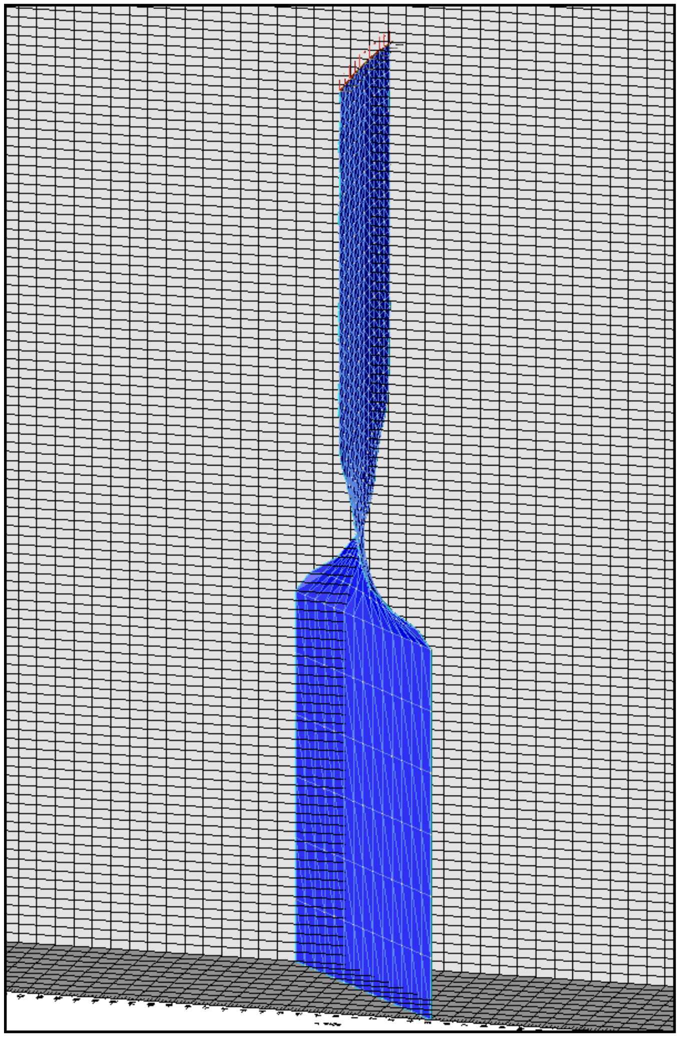
Available Methods – *hp*-GFEM/FEM

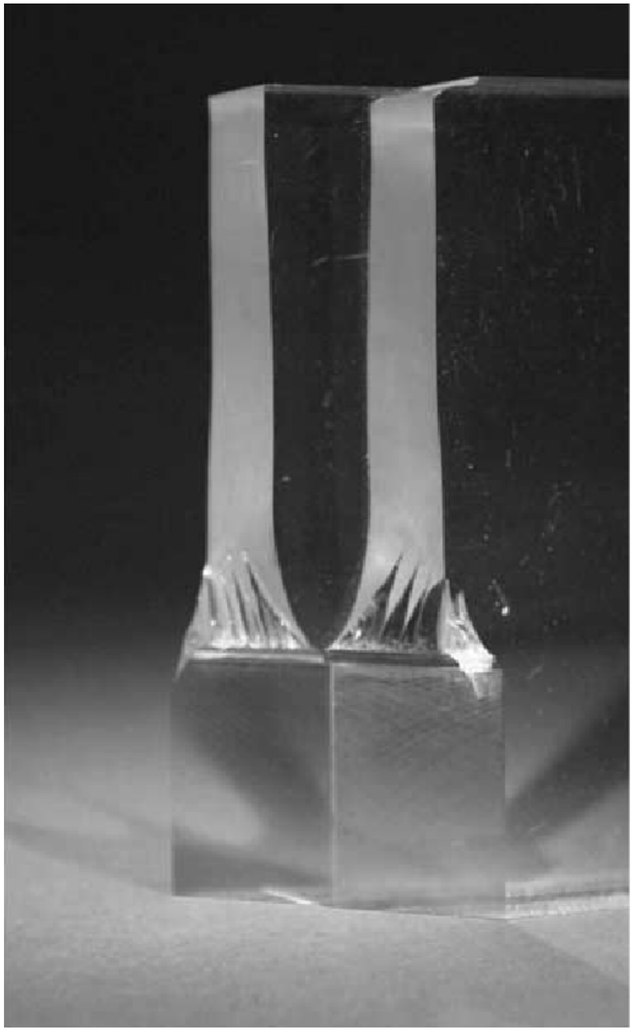


Two-Scale Generalized FEM – GFEM^{gl}

- Mesh with elements that are orders of magnitude larger than in a FEM mesh
- Fully compatible with FEM
- Single field formulation: Does not introduce stability (LBB) issues

Experimental Results





[Buchholz et al., 2004]

Computation of Solution at a Crack Step

$$\boldsymbol{u}_G = \underbrace{\tilde{\boldsymbol{u}}^0}_{\text{coarse scale (polynomial)}} + \underbrace{\boldsymbol{u}^{\text{gl}}}_{\text{fine scale (G-L)}} = [\boldsymbol{N}^0 \boldsymbol{N}^{\text{gl}}] \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^0 \\ \underline{\boldsymbol{u}}^{\text{gl}} \end{bmatrix}$$

$$\begin{aligned} \underline{\tilde{\boldsymbol{u}}}^0 &= \text{DOFs associate with coarse scale discretization} \\ \underline{\boldsymbol{u}}^{\text{gl}} &= \text{DOFs associate with G-L (hierarchical) enrichments} \end{aligned}$$

$$\text{dim}(\underline{\boldsymbol{u}}^{\text{gl}}) << \text{dim}(\underline{\tilde{\boldsymbol{u}}}^0)$$

This leads to

Computed by FEM code

$$\begin{bmatrix} \boldsymbol{K}^0 & \boldsymbol{K}^{0,\text{gl}} \\ \boldsymbol{K}^{\text{gl},0} & \boldsymbol{K}^{\text{gl}} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^0 \\ \underline{\boldsymbol{u}}^{\text{gl}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^0 \\ \boldsymbol{F}^{\text{gl}} \end{bmatrix}$$

Solve using, e.g., static condensation of $\underline{\boldsymbol{u}}^{\text{gl}}$

Computation of Solution at a Crack Step

From the first equation

$$\begin{aligned} \underline{\tilde{\boldsymbol{u}}}^{\,0} &= (\boldsymbol{K}^0)^{-1}\boldsymbol{F}^0 - (\boldsymbol{K}^0)^{-1}\boldsymbol{K}^{0,\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} \\ &= \underline{\boldsymbol{u}}^{\,0} - \boldsymbol{S}^{0,\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} \end{aligned}$$

Where

$$\boldsymbol{S}^{0,\mathrm{gl}} := (\boldsymbol{K}^0)^{-1}\boldsymbol{K}^{0,\mathrm{gl}}$$

$$\boxed{\boldsymbol{K}^0 \underbrace{\boldsymbol{S}^{0,\mathrm{gl}}}_{\text{pseudo coarse scale solutions}} = \underbrace{\boldsymbol{K}^{0,\mathrm{gl}}}_{\text{pseudo coarse scale loads}}}$$

$\boldsymbol{S}^{0,\mathrm{gl}}$ = Pseudo coarse scale solutions computed through forward and backward substitutions on $\boldsymbol{K}^0$

(by FEM code)

Computation of Solution at a Crack Step

From the second equation and the above

$$\boldsymbol{K}^{\text{gl}}\underline{\boldsymbol{u}}^{\text{gl}} = \boldsymbol{F}^{\text{gl}} - \boldsymbol{K}^{\text{gl},0}\left[\underline{\boldsymbol{u}}^{0} - \boldsymbol{S}^{0,\text{gl}}\underline{\boldsymbol{u}}^{\text{gl}}\right]$$

Thus

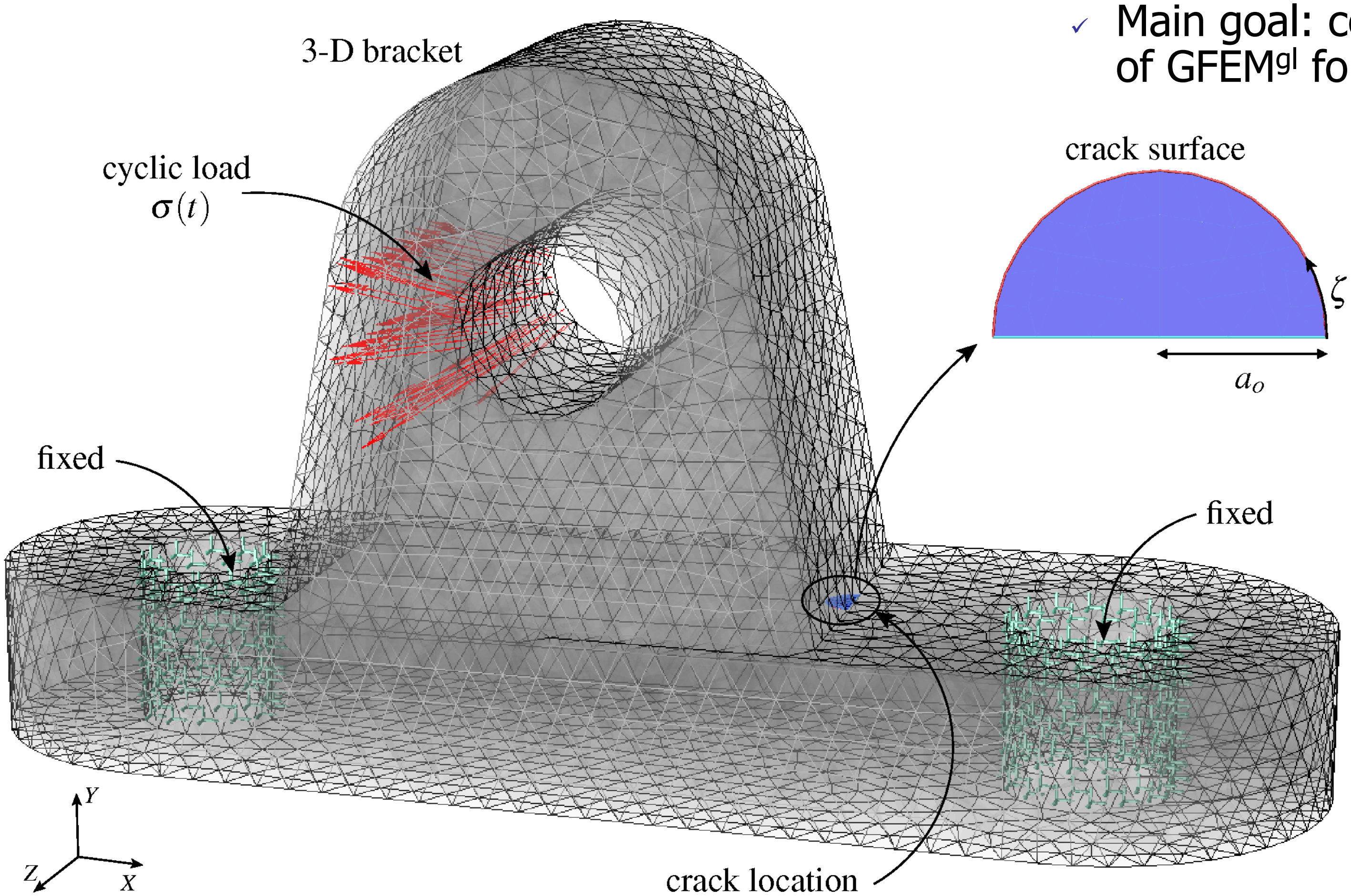
$$\underbrace{\left[\boldsymbol{K}^{\text{gl}} - \boldsymbol{K}^{\text{gl},0}\boldsymbol{S}^{0,\text{gl}}\right]}_{\widehat{\boldsymbol{K}}^{\text{gl}}}\underline{\boldsymbol{u}}^{\text{gl}} = \underbrace{\boldsymbol{F}^{\text{gl}} - \boldsymbol{K}^{\text{gl},0}\underline{\boldsymbol{u}}^{0}}_{\widehat{\boldsymbol{F}}^{\text{gl}}}$$

$$\boxed{\begin{gathered}\widehat{\boldsymbol{K}}^{\text{gl}}\underline{\boldsymbol{u}}^{\text{gl}} = \widehat{\boldsymbol{F}}^{\text{gl}}\\ \underline{\tilde{\boldsymbol{u}}}^{0} = \underline{\boldsymbol{u}}^{0} - \boldsymbol{S}^{0,\text{gl}}\underline{\boldsymbol{u}}^{\text{gl}}\\ \boldsymbol{u}_E = \tilde{\boldsymbol{u}}^{0} + \boldsymbol{u}^{\text{gl}} = \left[\boldsymbol{N}^{0}\boldsymbol{N}^{\text{gl}}\right]\begin{bmatrix}\underline{\tilde{\boldsymbol{u}}}^{0}\\ \underline{\boldsymbol{u}}^{\text{gl}}\end{bmatrix}\end{gathered}}$$

Computation of $\boldsymbol{u}_G$ involves forward- and back-substitutions on $\boldsymbol{K}^0$

Computational Efficiency

- **Bracket with half-penny shaped crack**



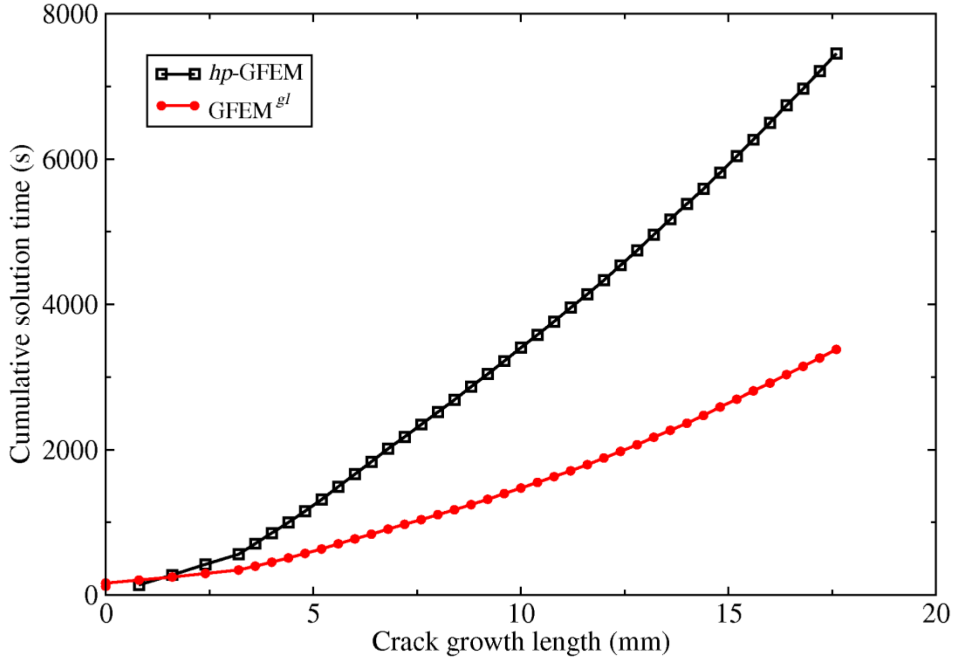
3-D mesh courtesy of Altair Engineering

- ✓ *hp*-GFEM as reference solution
- ✓ Main goal: computational efficiency of $GFEM^{gl}$ for crack growth

Movie

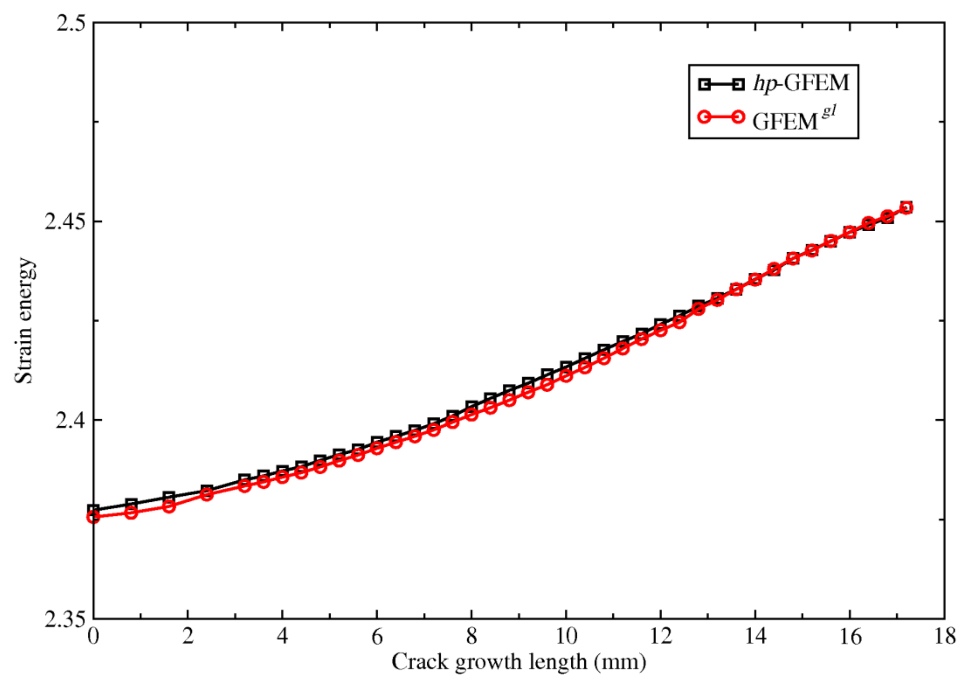
Computational Efficiency

- **Computational cost analysis**



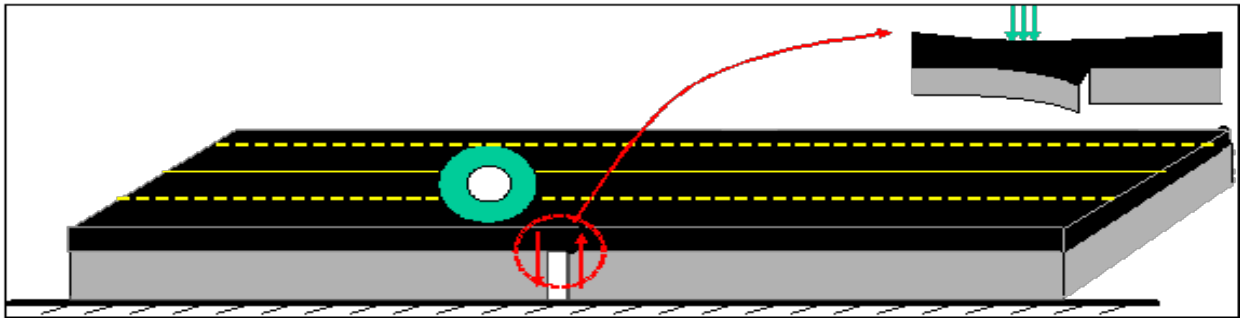
- GFEM^{gl}:
  115,470 + 27 *dofs* (min)
  115,470 + 84 *dofs* (max)
- *hp*-GFEM:
  186,666 global *dofs* (min)
  255,618 global *dofs* (max)

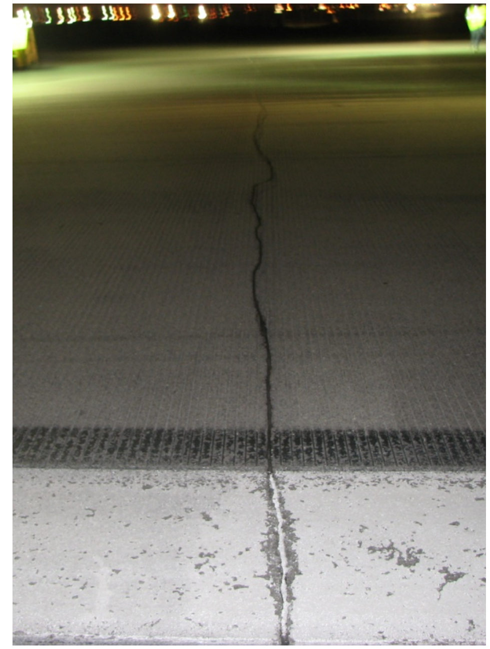
- **Strain Energy**



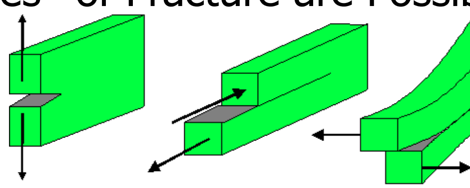
- ~ 60% computational cost reduction
- *hp*-GFEM and GFEM^{gl} solutions show good agreement

Reflective Crack Growth in Airfield Pavements



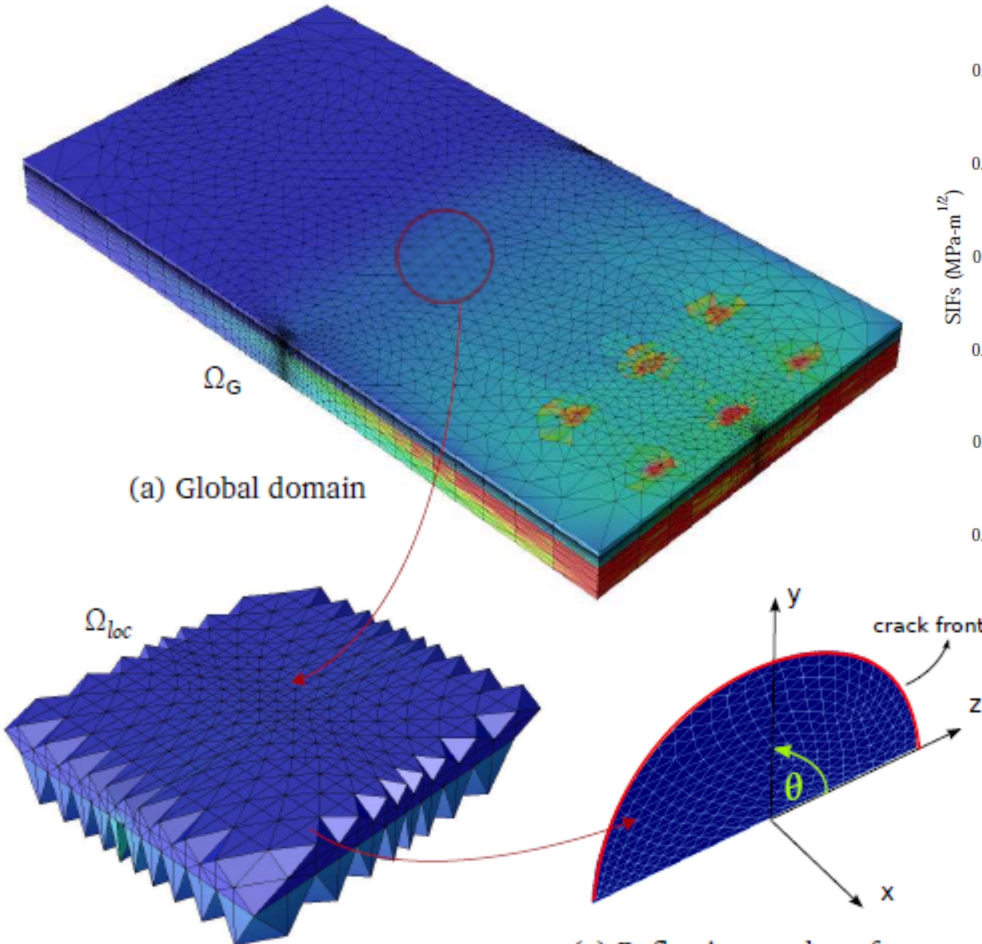


- Cracks and Joints in Underlying Pavement "Reflect" up to the Surface due to Stress Concentration Effect
- Three "Modes" of Fracture are Possible:

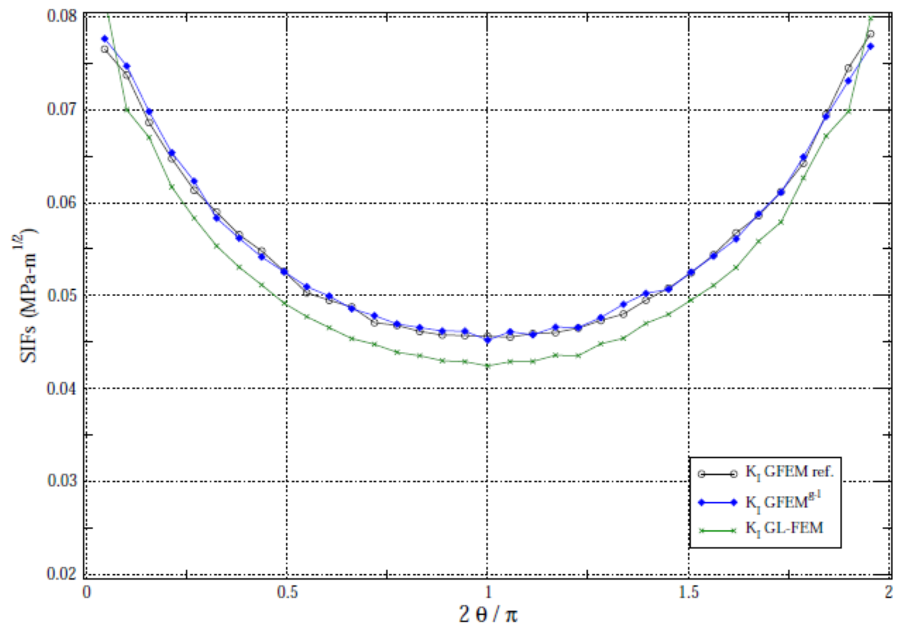


- **Objectives**
  - Providing better understanding of RC in airfield pavements
  - Being used to assist in development of reflective cracking test at National Airport Pavement Test Facility (NAPTF)

Reflective Crack on Airfield Pavement



(a) Global domain
(b) Local domain
(c) Reflective crack surface



Movie

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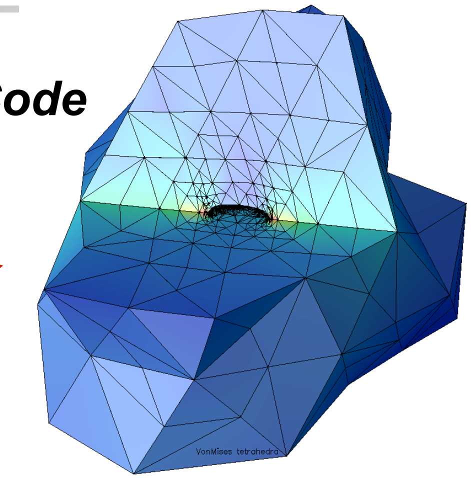
Non-Intrusive Implementation in Existing FEM Codes

- Different solvers can be used to solve coarse and fine-scale problems

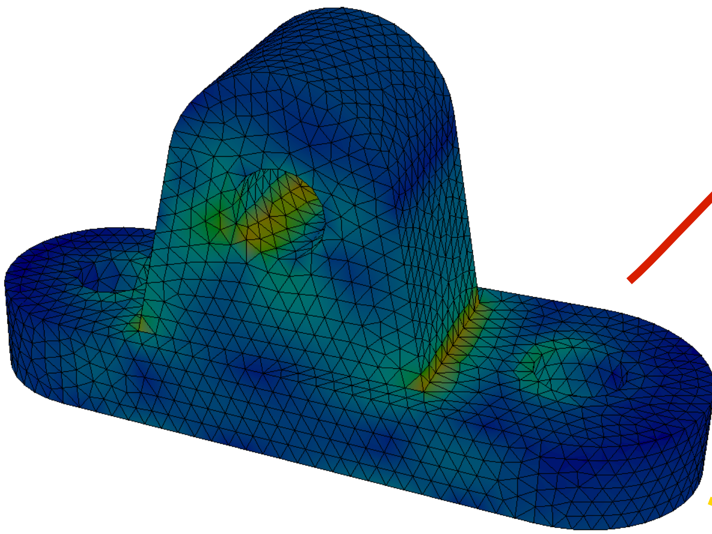
GFEM Code



Local solution
Global-local enrichment

Commercial FEM Code

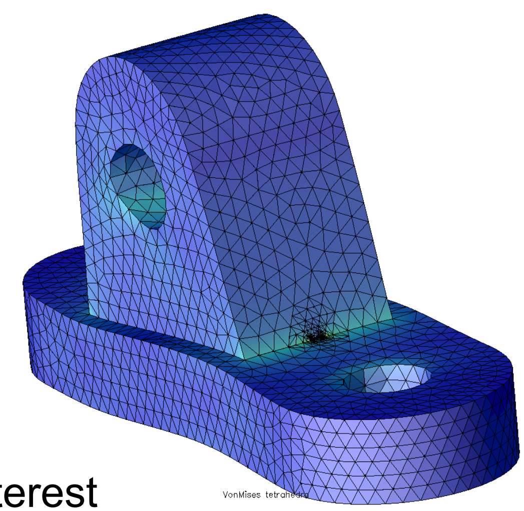
Global solution
BC for local prob.



Pseudo loads

Pseudo solutions

Initial global solution
Uncracked domain



Enriched global solution
Compute quantities of interest

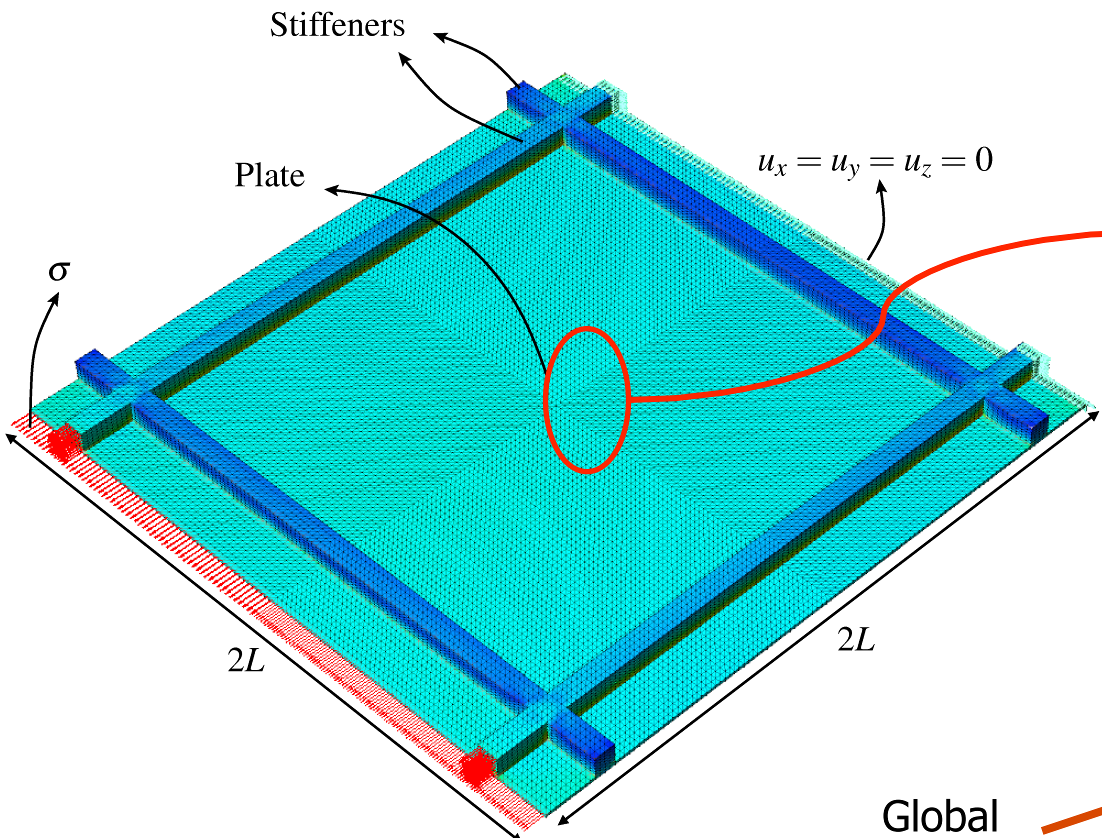
*with J. Pereira, P. Gupta, J. Plews and T. Eason

Related Methods

- Krause R, Rank, E. Multiscale computations with a combination of the h- and p-versions of the finite-element method. CMAME, 2003
- Bordas S, Moran B. Enriched finite elements and level sets for damage tolerance assessment of complex structures, EFM, 2006
- Gendre L, Allix O, Gosselet P, Comte F. Non-intrusive and exact global/local techniques for structural problems with local plasticity. CM, 2009
- Gendre L, Allix O, Gosselet P. A two-scale approximation of the Schur complement and its use for non-intrusive coupling. IJNME, 2011

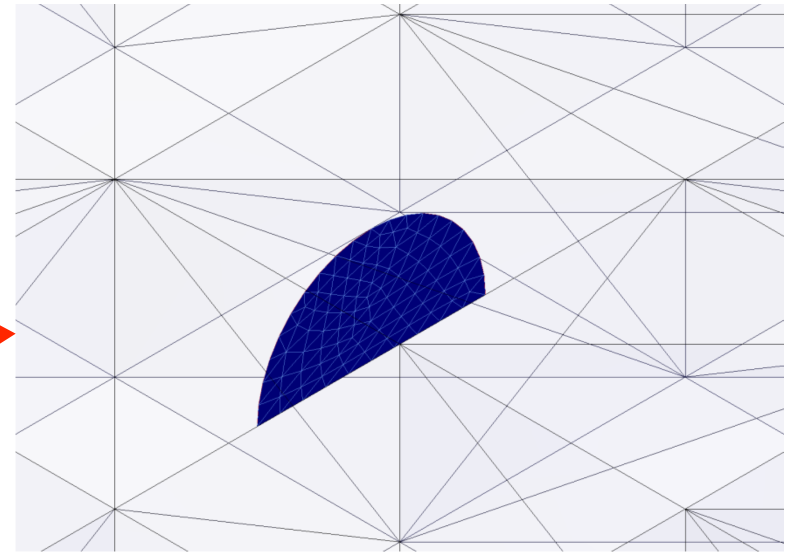
Stiffened Panel with Surface Crack*

- Global Problem in Abaqus



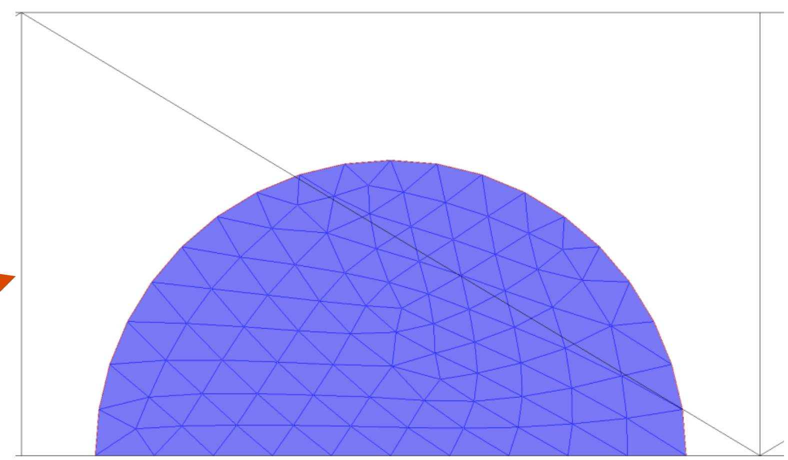
$L = 300\,mm$ $t = 3\,mm$

$b = 20\,mm$ $d = 50\,mm$



Crack radius r = 2mm

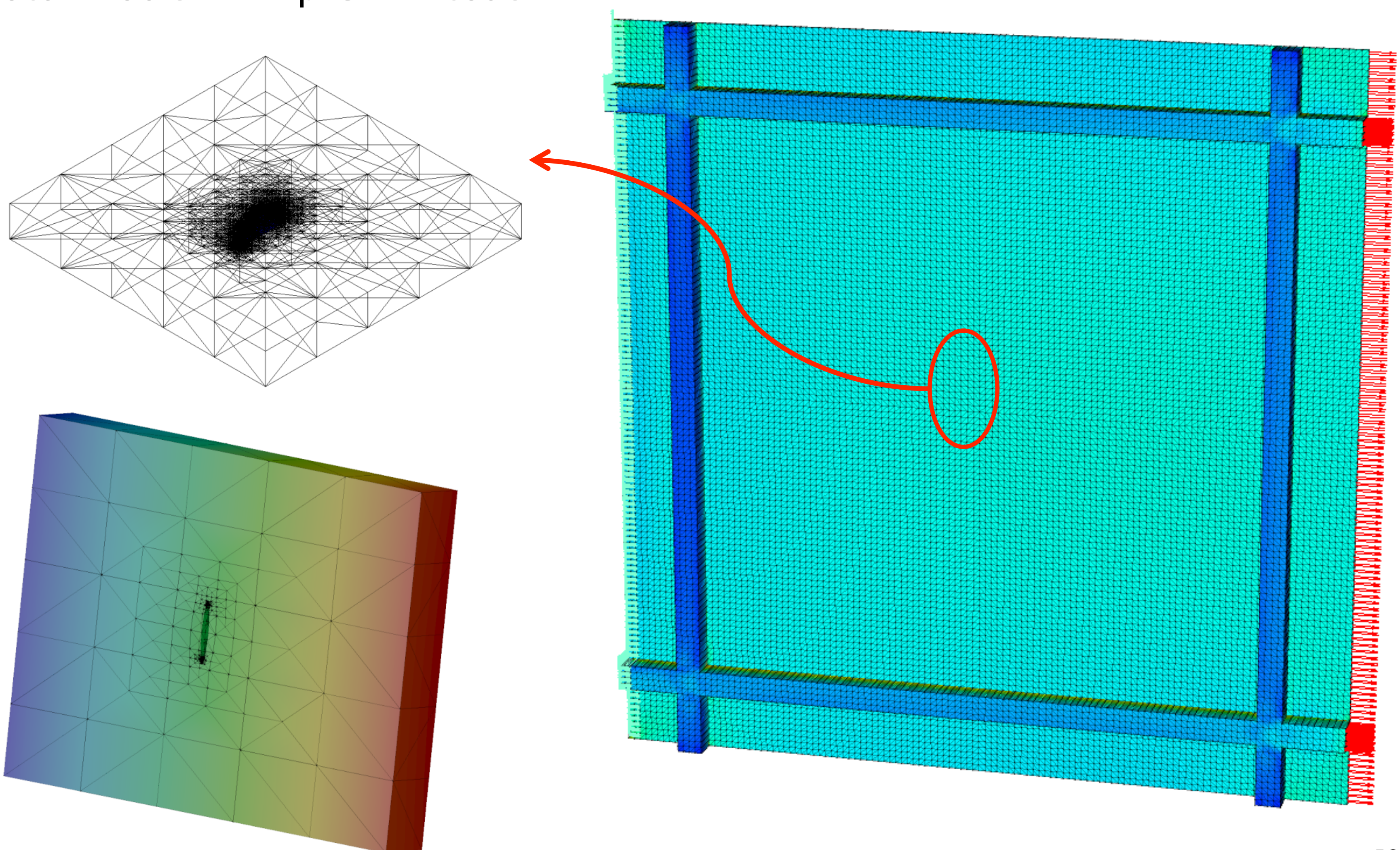
Global TET10 in Abaqus



*with P. Gupta, J. Pereira and T. Eason

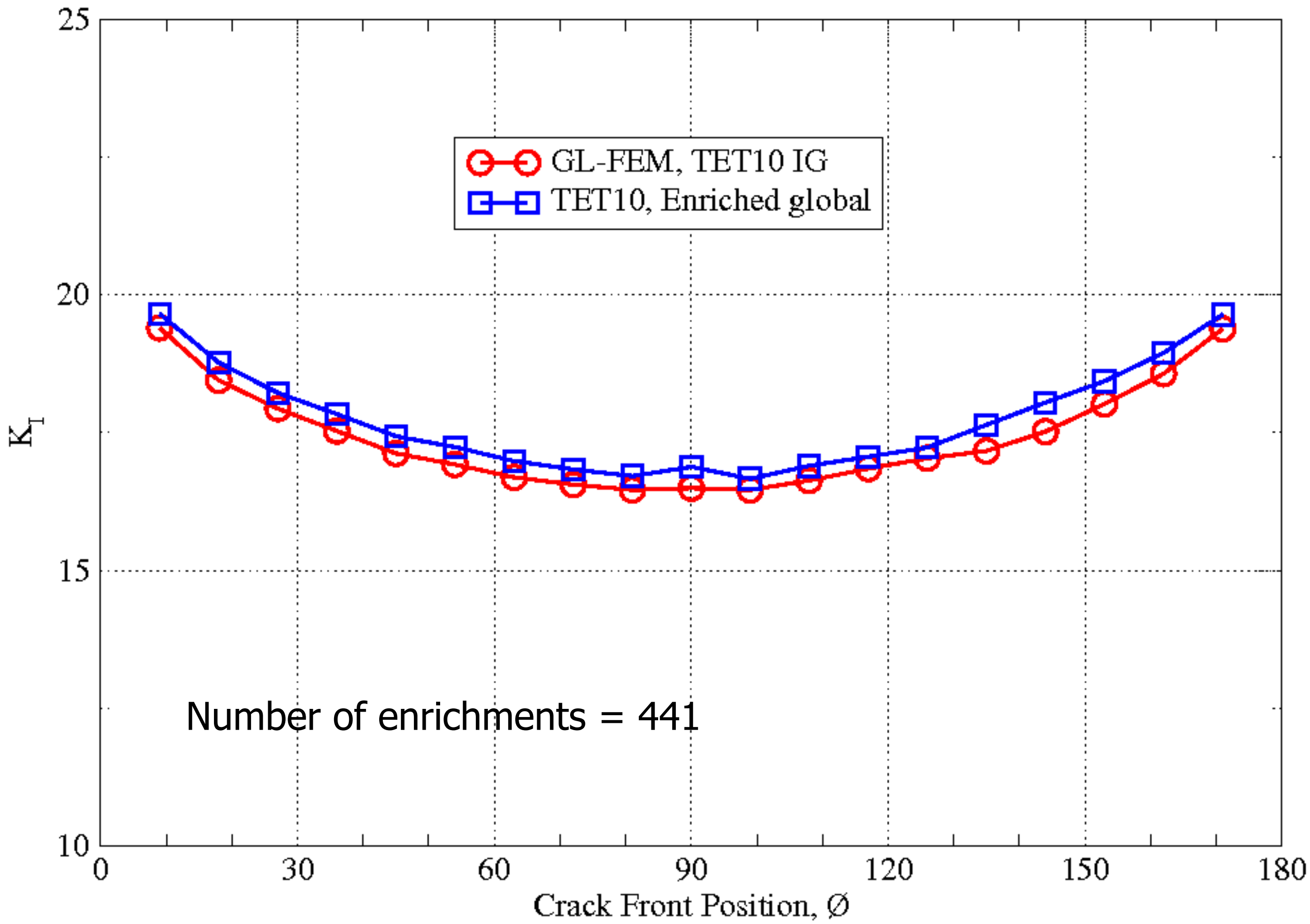
Stiffened Panel with Surface Crack

- Local Problem in hp-GFEM code



Stiffened Panel with Surface Crack

Mode-I SIF along crack front



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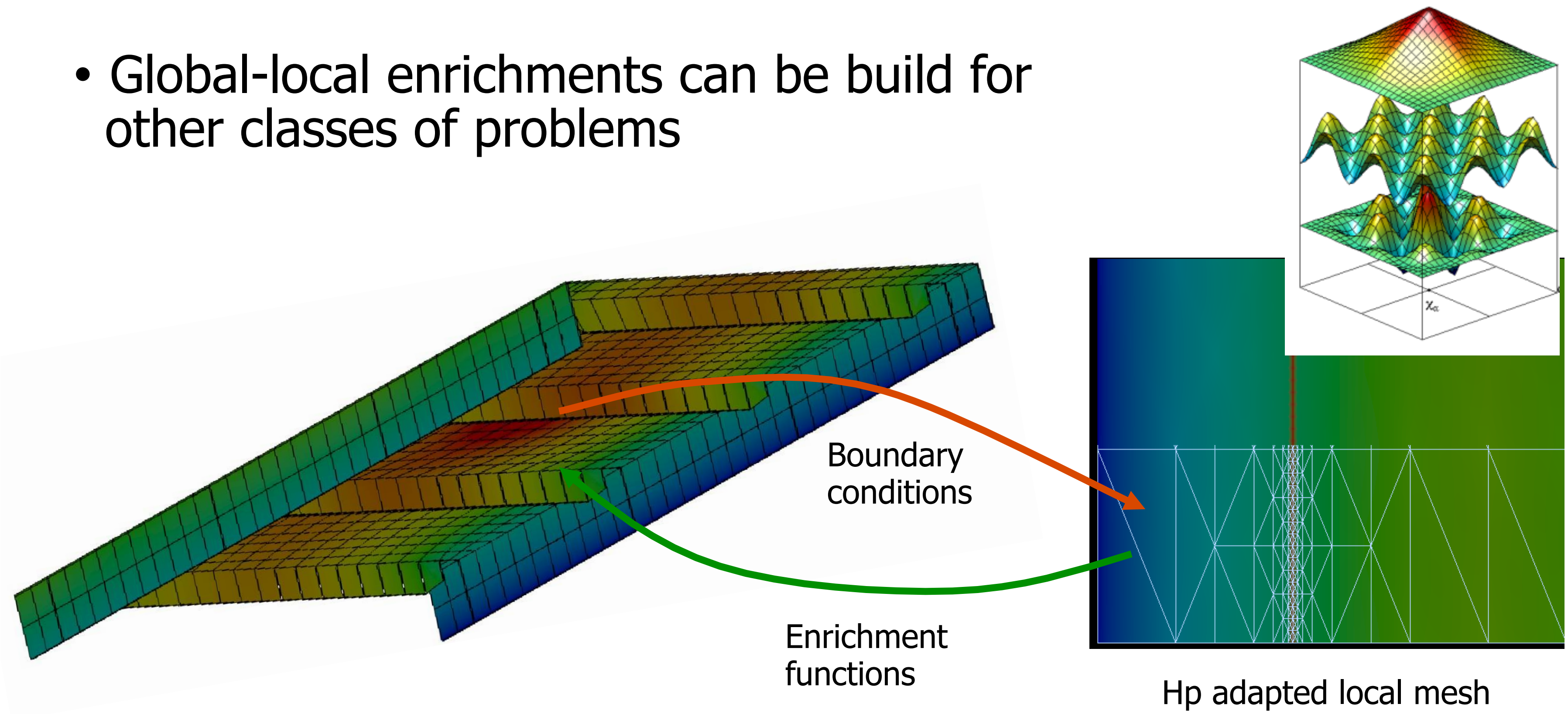
3-D Problems with Sharp Thermal Gradients*

- Global-local enrichments can be build for other classes of problems



Hp adapted local mesh

Initial/Enriched global problem

Enrichment of global FEM discretization with local solution:

$$\boxed{\phi_\alpha = \varphi_\alpha u_{loc}}$$

* with P. O'Hara and T. Eason

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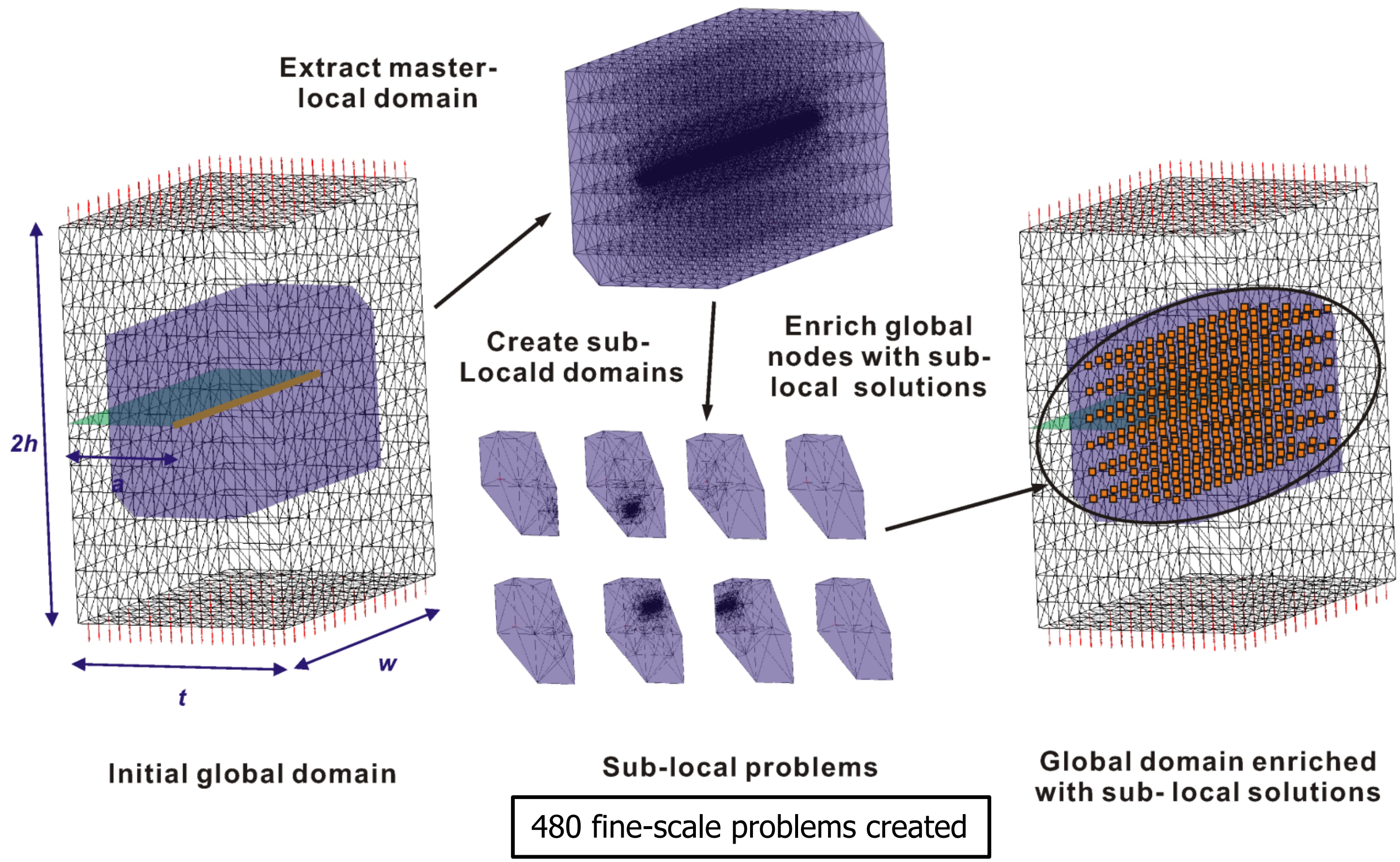
Parallel Computation of Enrichment Functions *

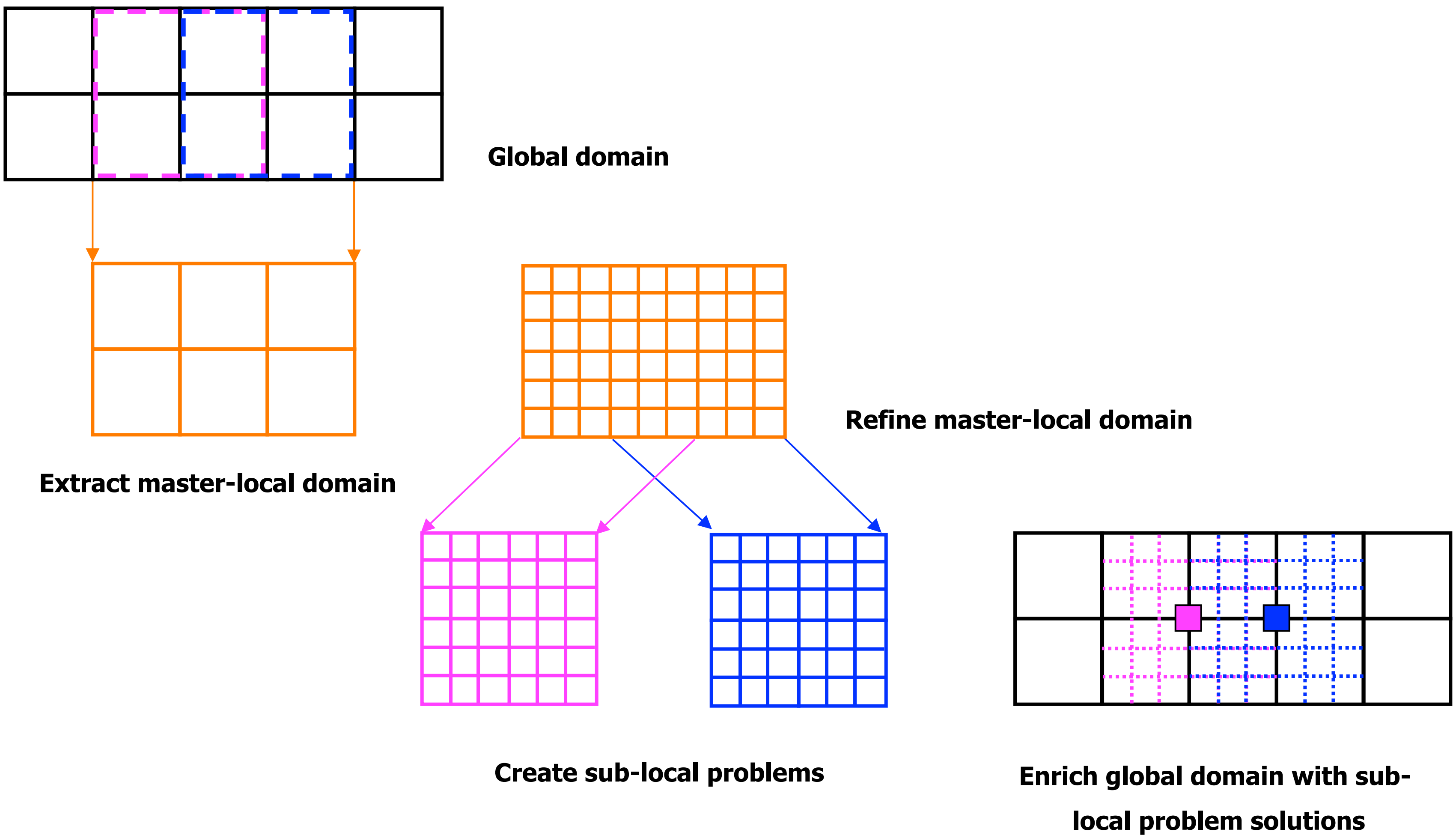
- A large number of small fine-scale problems can be created instead of a single one
- *No communication is involved in their parallel solution*



*with D.-J. Kim and N. Sohb

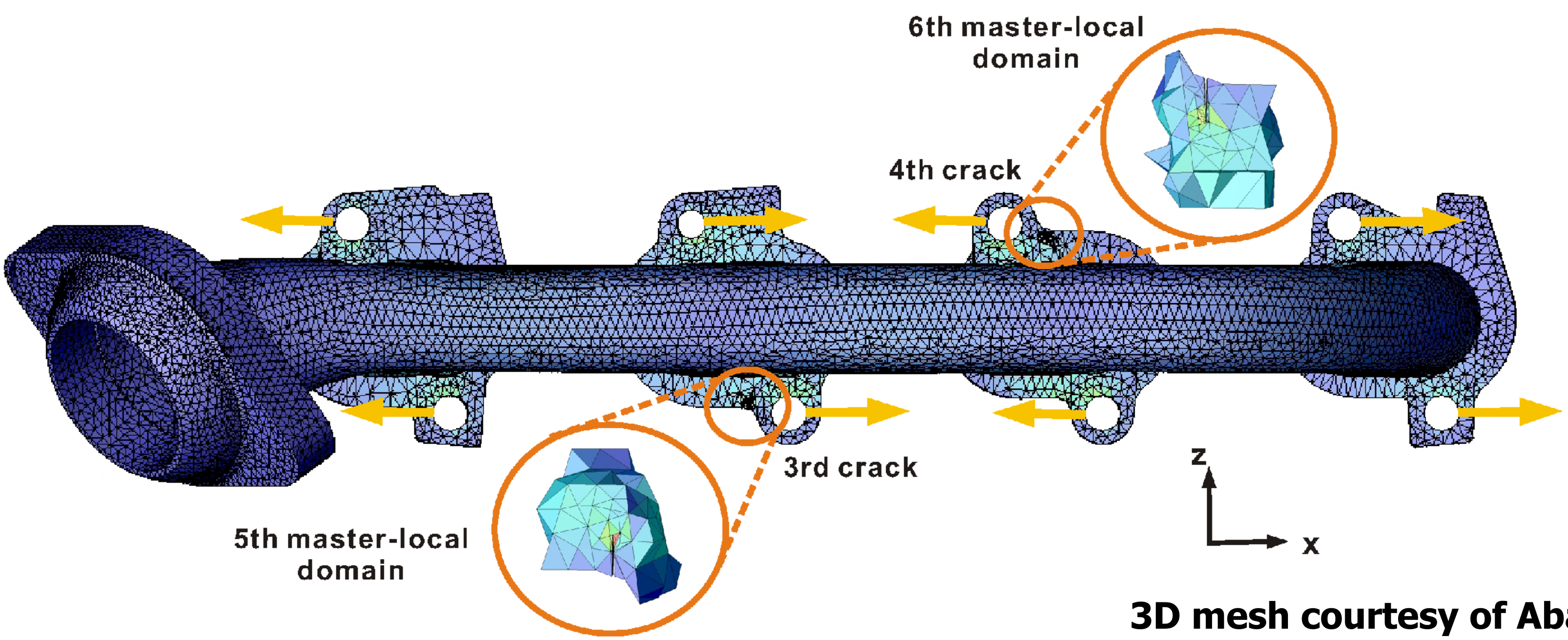
Master-Sub Local Problem Approach

- **KEY IDEA: Subdivide large local domains into smaller ones while keeping compatibility between local meshes.**



Mechanical Manifold with Multiple Cracks

- Eight crack fronts and master-local problems
- 983 sub-local problems solved in parallel
- Comparable DNS model has 1,605,960 dofs

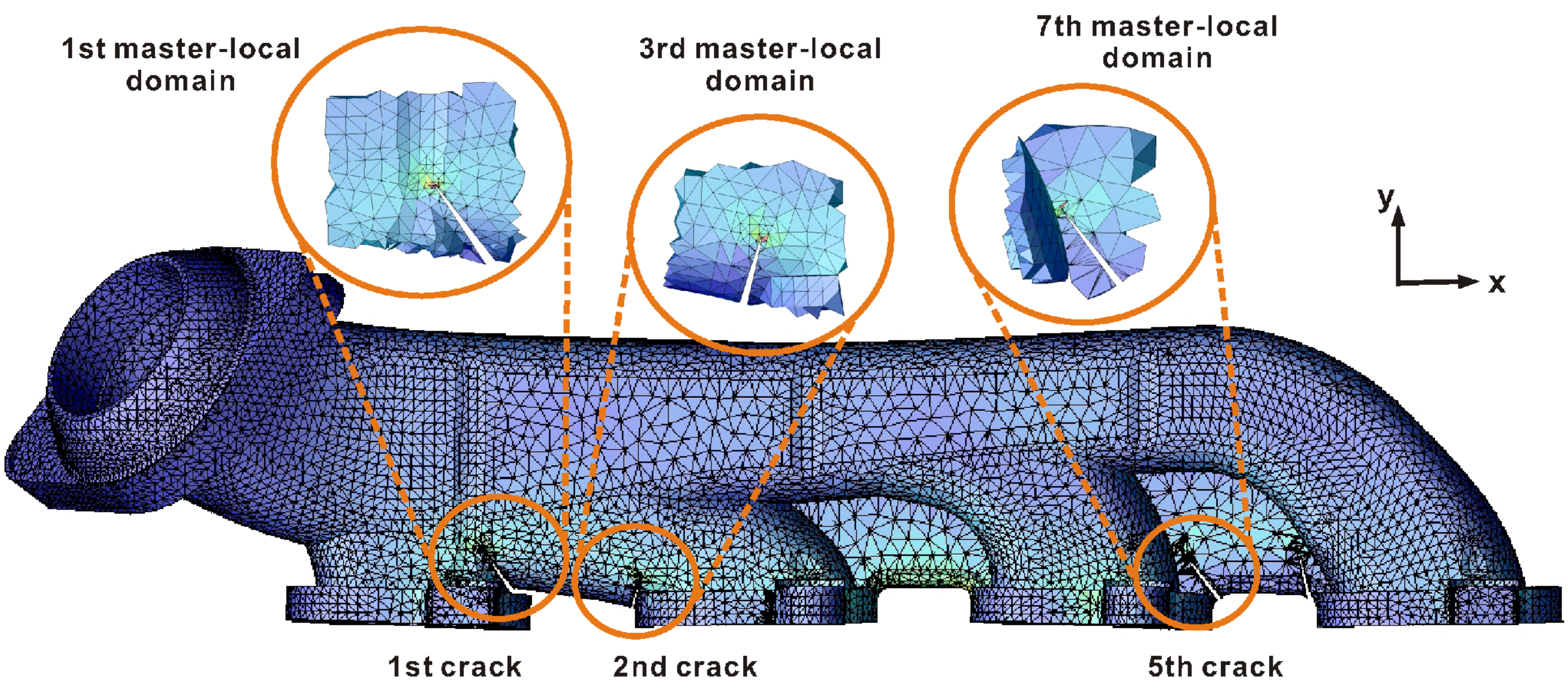


3D mesh courtesy of Abaqus

Top View and Boundary Conditions

Mechanical Manifold with Multiple Cracks

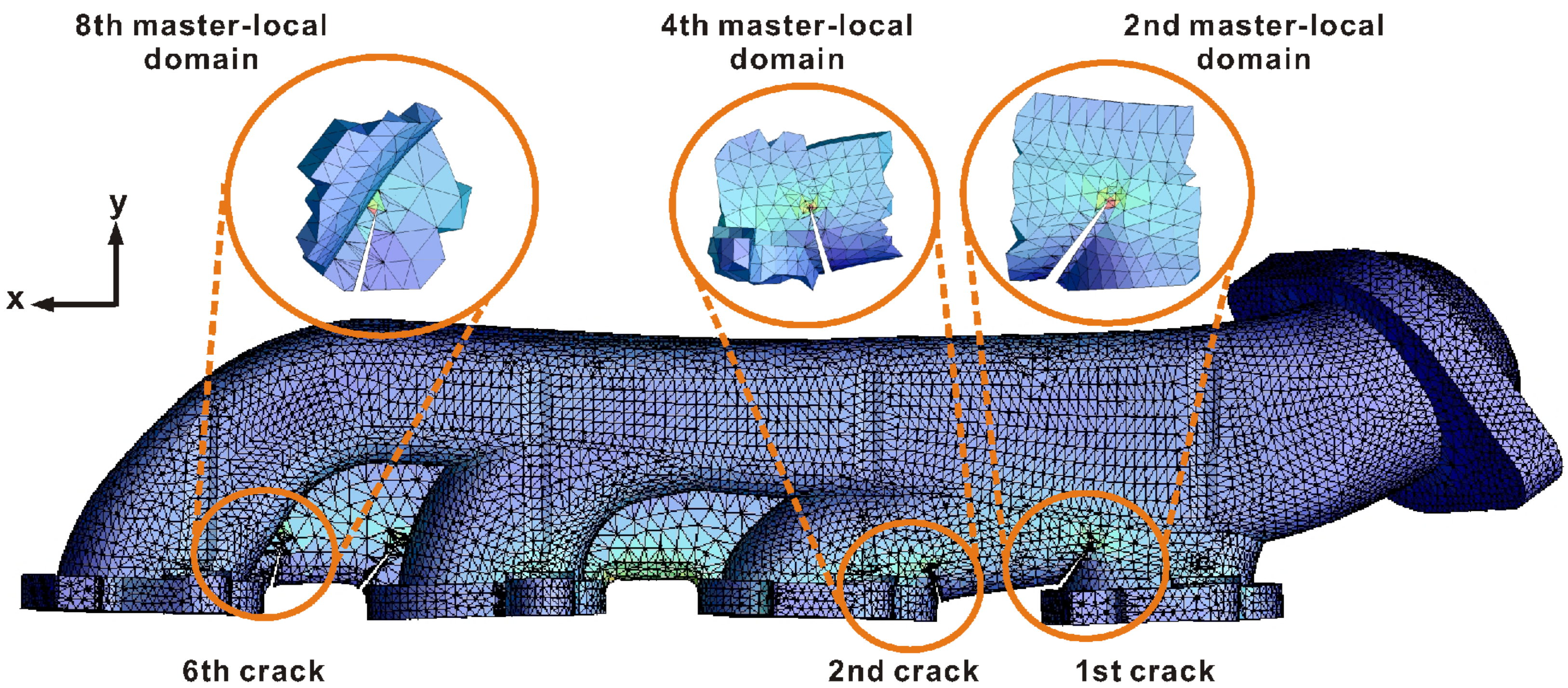
- Eight crack fronts and master-local problems
- 983 sub-local problems solved in parallel



Front View

Mechanical Manifold with Multiple Cracks

- Eight crack fronts and master-local problems
- 983 sub-local problems solved in parallel

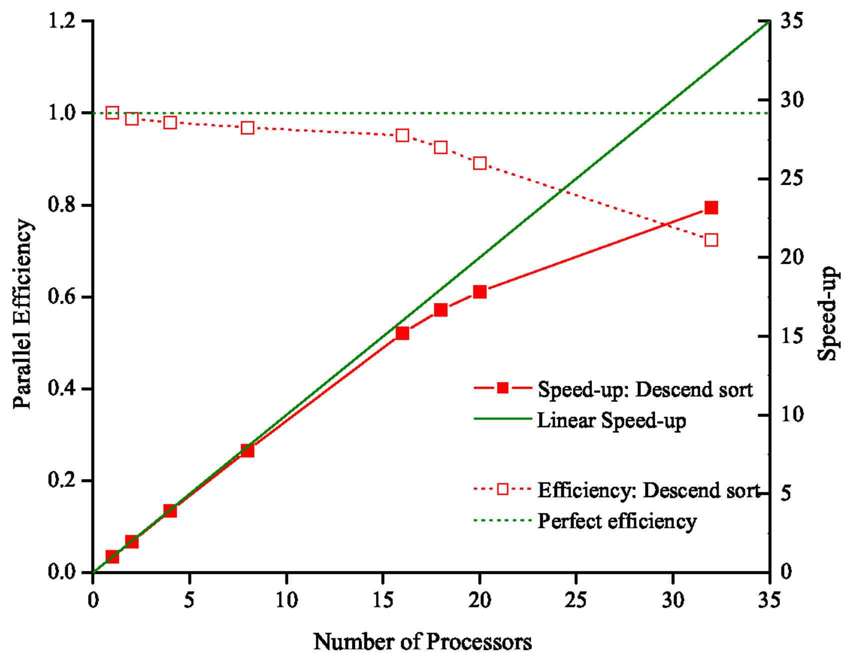


Back View

Mechanical Manifold with Multiple Cracks

- **Parallel performance on a shared memory machine (NUMA architecture)**

Number of processors	CPU time (s)	Parallel efficiency	Speed-up
1	1537.4	N/A	N/A
2	778.1	0.988	1.976
4	392.1	0.980	3.921
8	198.3	0.969	7.752
16	101.1	0.951	15.214
18	92.3	0.926	16.666
20	86.3	0.891	17.823
32	66.4	0.724	23.161



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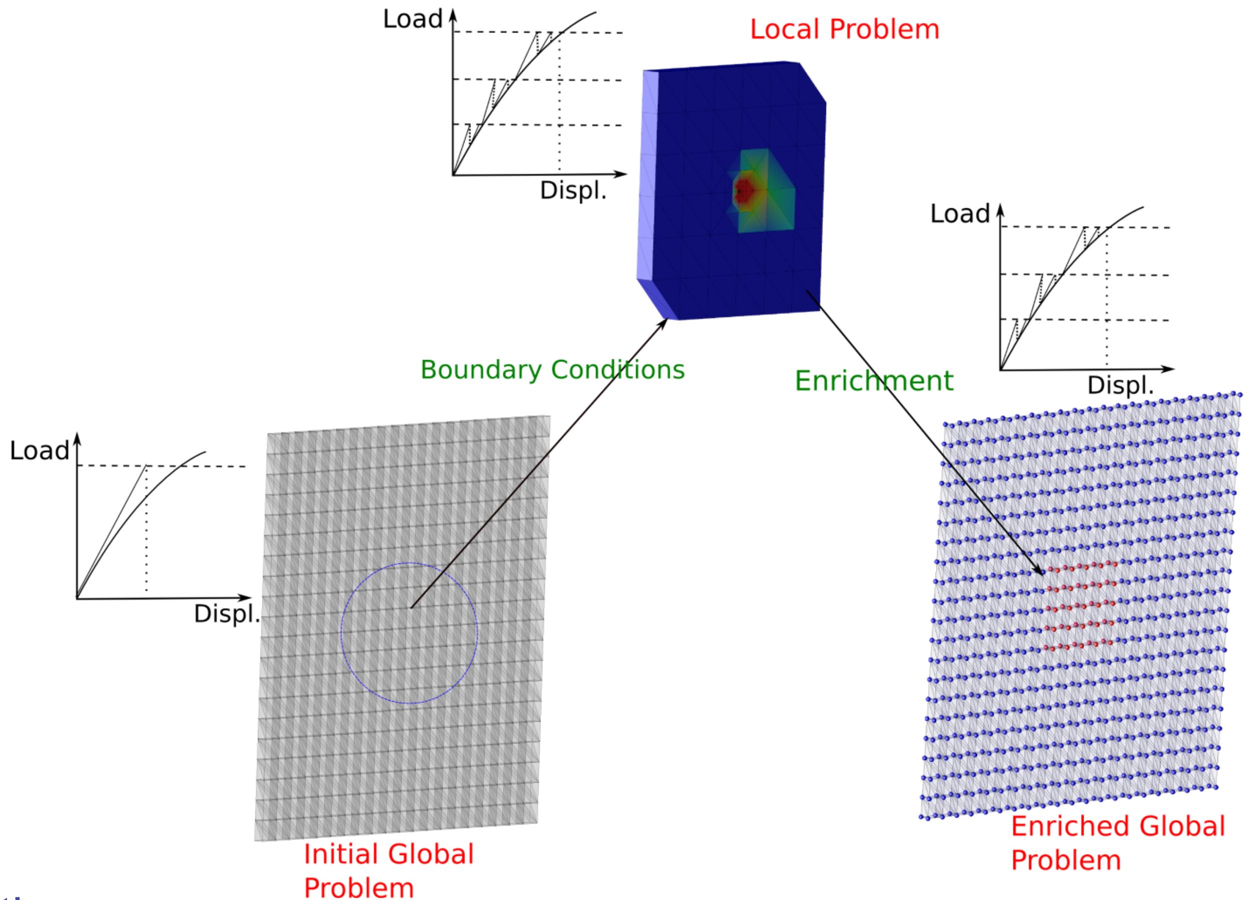
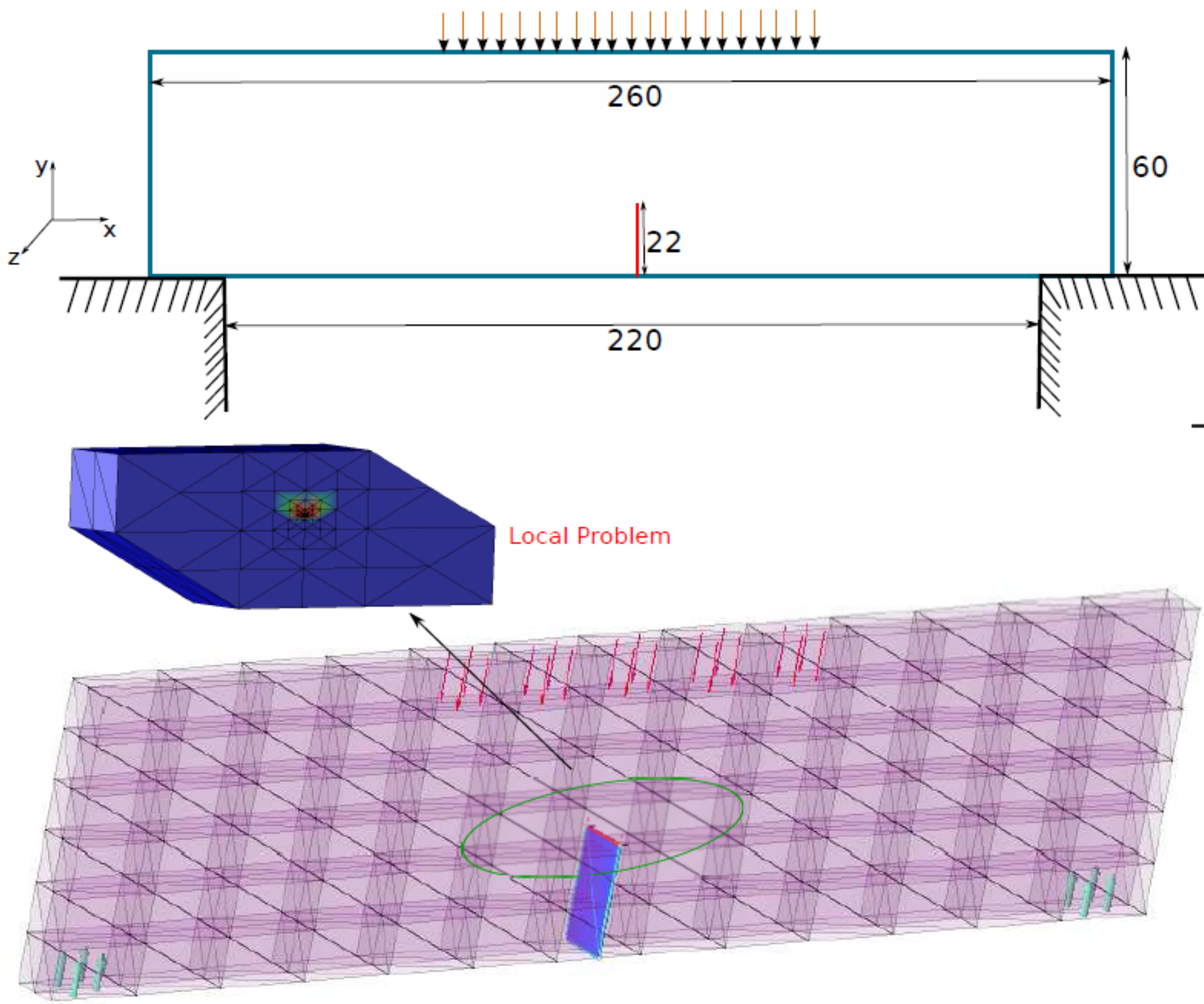
GFEM^{gl} for Nonlinear Fracture Mechanics



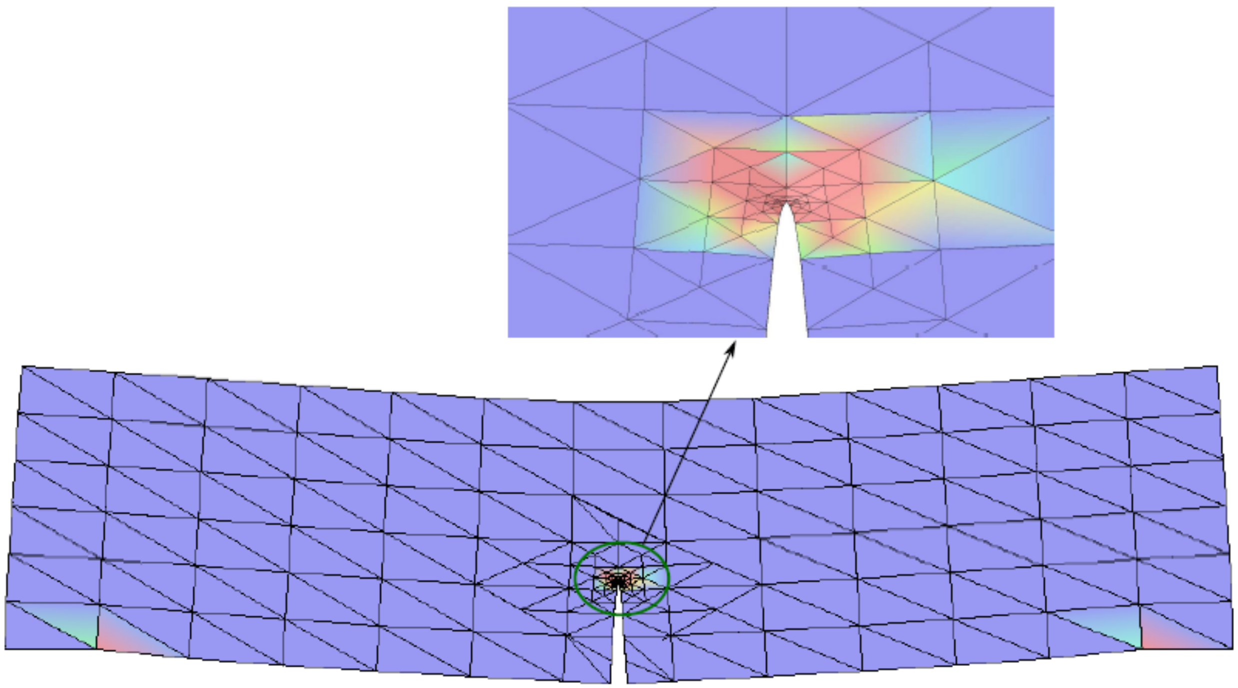
- **Key Properties:**
- Uses available information at a simulation step to build approximation spaces for the next step
- Uses coarse FEM meshes; solution spaces of much reduced dimension than in the FEM
- Two-way information transfer between scales; account for interactions among scales

3-D Beam with a Crack



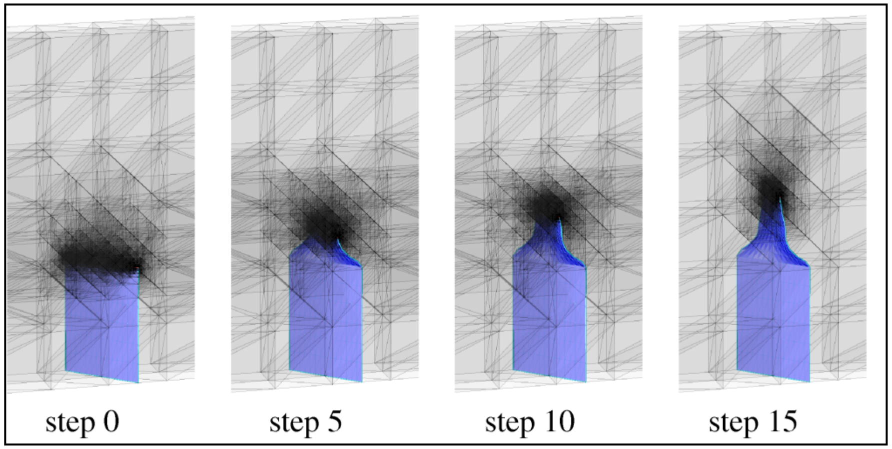
Method	Number of DOFs
GFEM^{gl}	9,216
hp-GFEM	110,676

3-D Beam with a Crack

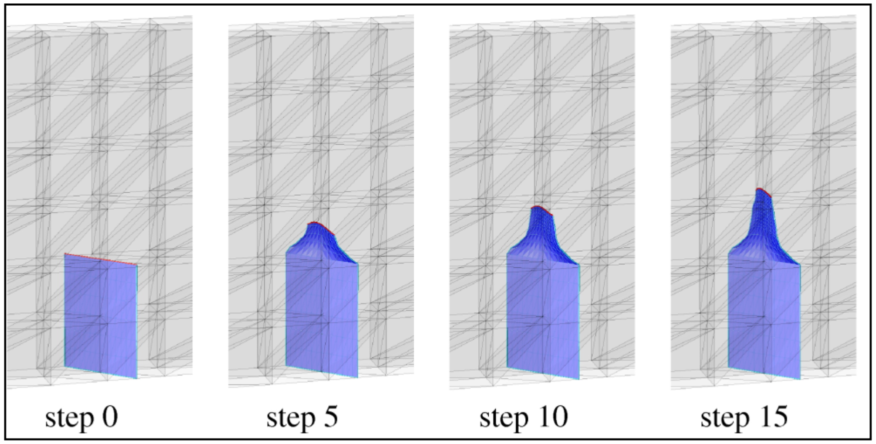


Load step	Number of iterations	
	GFEM[gl]	*hp-GFEM*
1	2	1
2	2	1
3	2	2
4	2	3
5	2	3
6	3	3
7	3	4
8	3	5
9	3	4
10	3	4
11	3	7
12	3	5
13	3	4
14	3	4
15	3	8
16	3	9
17	3	4
18	3	4
19	3	4
20	4	4

Concluding Remarks

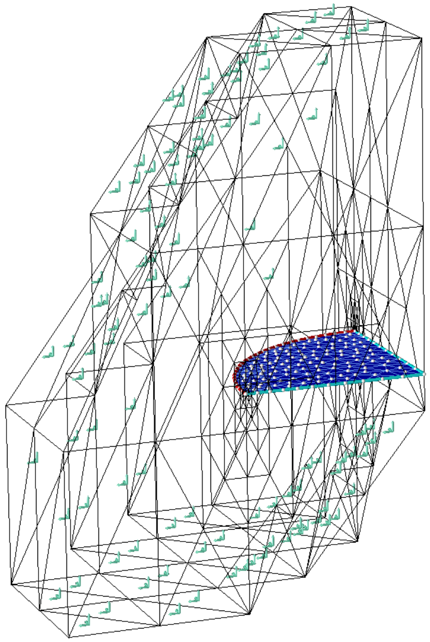


Available methods require AMR



Multiscale Generalized FEM

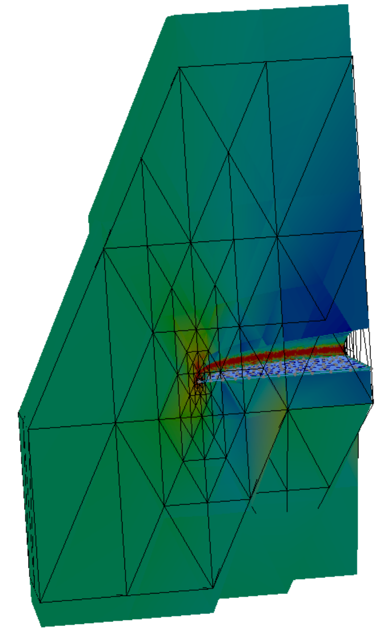
- FAST: Coarse-scale model of much reduced dimension than FEM; Fine-Scale computations are intrinsically parallelizable; recycle coarse scale solution
- ACCURATE: Can deliver same accuracy as adaptive mesh refinement (AMR) on meshes with elements that are orders of magnitude larger than in the FEM
- STABLE: Uses single-field variational principles
- TRANSITION: Fully compatible with FEM



Questions?

caduarte@uiuc.edu

http://netfiles.uiuc.edu/caduarte/www/



Support: