Generalized finite element approaches for analysis of localized non-linear thermo-mechanical effects

To the memory of Prof. Ted Belytschko

C. Armando Duarte and Julia Plews
Dept. of Civil and Environmental Engineering
Computational Science and Engineering

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Motivation: Multiscale Structural Analysis

- Thermo, mechanical and acoustic loads on hypersonic aircrafts lead to highly localized non-linear stress fields: 3-D finite element models with fine meshes are required.

- Highly localized non-linear 3-D effects
- Most of the structure remains linear elastic
- 3-D FEM: Large aspect ratio of elements may lead to numerical instabilities.

Representative hypersonic skin panel
[Sobotka et al., 2013]
Motivation: Multiscale Structural Analysis

- Hypersonic aircraft panels are assembled from sub-components using hundreds of fasteners or spot welds.
- Multiple spatial scales: Skin panel, stiffeners, spot welds.

Panel 1 to Sub-Structure Attachment
[AFRL-RB-WP-TR-2012-0280]
Motivation: Multiscale Structural Analysis

- Representation of a spot weld in the FEM requires detailed meshing.
- Hundreds of spot welds in one single panel: Not feasible to mesh them all.
- Multi-point constraint is used instead in the industry: This leads to mesh dependent solutions even far from spot welds!

**Strategy:**

- Formulate a two-scale GFEM for this class of problems;
- Keep global mesh coarse and resolve spot welds through enrichment functions computed in parallel.
Outline

- Motivation
- Generalized finite element methods: Basic ideas
- Bridging scales with GFEM:
  - Global-local enrichments for heat equation and nonlinear thermo-mechanical problems
- Numerical examples
- Conclusions
Generalized Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

\[
\mathcal{S}_{GFEM} = \mathcal{S}_{FEM} + \mathcal{S}_{ENR}
\]

Low order FEM space \hspace{1cm} Enrichment space with functions related to the given problem

\[
\mathcal{S}_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R}
\]

\[
\mathcal{S}_{ENR} = \sum_{\alpha \in I^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}
\]

\[
L_{\alpha i} \in \chi_\alpha(\omega_\alpha)
\]

Enrichment function \hspace{1cm} Patch space
Generalized Finite Element Method

\[ S_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha} \]

\[ \phi_{\alpha i}(x) = \varphi_\alpha(x) L_{\alpha i}(x) \quad \sum_{\alpha} \varphi_\alpha(x) = 1 \]

- Allows construction of shape functions incorporating a-priori knowledge about solution

[Oden, Duarte & Zienkiewicz, 1996]
Bridging Scales with Global-Local Enrichment Functions*

- Enrichment functions computed from solution of local boundary value problems: Global-Local enrichment functions

- **Idea:** Use available numerical solution at a simulation step to build shape functions for next step (quasi-static, transient, non-linear, etc.)

- Enrichment functions are produced numerically on-the-fly through a global-local analysis

- Use a *coarse* mesh enriched with Global-Local (GL) functions

- \( \text{GFEM}^{\text{gl}} = \text{GFEM with global-local enrichments} \)

* [Duarte et al. 2005]
Global-Local Enrichments for Heat Equation

\[ \rho c \frac{\partial u}{\partial t} = \nabla (\kappa(\mathbf{x}) \nabla u) + Q(\mathbf{x}, t) \quad \text{in} \quad \Omega \]

where \( u(\mathbf{x}, t) \) is the temperature field, \( \rho c \) is the volumetric heat capacity and \( Q(\mathbf{x}, t) \) is the internal heat source. \( \kappa(\mathbf{x}) \) may be oscillatory.

\[ -\kappa \frac{\partial u}{\partial n} = \eta (\bar{u} - u) \quad \text{on} \quad \Gamma_c \]

\[ -\kappa \frac{\partial u}{\partial n} = \bar{f} \quad \text{on} \quad \Gamma_f \]

\[ u(\mathbf{x}, 0) = u^0(\mathbf{x}) \quad \text{at} \quad t^0 \]

where \( u^0(\mathbf{x}) \) is the prescribed temperature field at time \( t = t^0 \)
Domain Subjected to Sharp Laser Flux

- **Goal:** Solve with GFEM\textsuperscript{gl} on the mesh shown below

\[
\bar{f}(\mathbf{x}, t) = I_0 \cdot f(t) \cdot \frac{1}{2\pi a^2} \cdot G(\mathbf{x}, b, a)
\]
\[
f(t) = 1 - \exp(-\gamma \cdot t)
\]
\[
G(\mathbf{x}, b, a) = \exp\left(\frac{-(x - b)^2}{2a^2}\right)
\]

Laser flux:

- Sharp (Gaussian), localized heat flux applied as shown

Convection BCs applied everywhere else

Local material heterogeneity:

\[
\kappa_a = 50 \kappa_b
\]
Global-Local Enrichments for Heat Equation

Let \( u^n_G(x) \in S^{GFEM,n}_G(\Omega) \) be the GFEM solution at time \( t = t^n = n\Delta t \)

Find \( u^n_G \in S^{GFEM,n}_G(\Omega_G) \) such that, \( \forall \ w^n_G \in S^{GFEM,n}_G(\Omega_G) \)

\[
\frac{\rho c}{\Delta t} \int_{\Omega} w^n_G u^n_G \, d\Omega + \int_{\Omega} (\nabla w^n_G)^T \kappa \nabla u^n_G \, d\Omega + \eta \int_{\Gamma_c} w^n_G u^n_G \, d\Gamma = 0
\]

\[
\frac{\rho c}{\Delta t} \int_{\Omega} w^n_G u^{n-1}_G \, d\Omega + \int_{\Gamma_f} f^n w^n_G \, d\Gamma + \eta \int_{\Gamma_c} \bar{u}^n w^n_G \, d\Gamma + \int_{\Omega} Q^n w^n_G \, d\Omega
\]
Global-Local Enrichments for Heat Equation

Let $u^n_G(x) \in S^{GFEM,n}_G(\Omega)$ be the GFEM solution at time $t = t^n = n\Delta t$

- Define local domain around the laser flux location at time $t = t^{n+1}$
Global-Local Enrichments for Heat Equation

- Solve following local problem at time $t = t^{n+1}$ using, e.g., $hp$-GFEM

Find $u_{L}^{n+1} \in S_{L}^{GFEM,n+1}(\Omega_{L})$ such that, $\forall w_{L}^{n+1} \in S_{L}^{GFEM,n+1}(\Omega_{L})$

$$
\int_{\Omega_{L}} (\nabla w_{L}^{n+1})^{T} \kappa \nabla u_{L}^{n+1} d\Omega + \eta \int_{\partial\Omega_{L} \setminus (\partial\Omega_{L} \cap \Gamma_{f})} w_{L}^{n+1} u_{L}^{n+1} d\Gamma \\
= \int_{\Omega_{L}} \bar{Q}^{n+1} w_{L}^{n+1} d\Omega + \int_{\partial\Omega_{L} \cap \Gamma_{f}} \bar{f}^{n+1} w_{L}^{n+1} d\Gamma \\
+ \eta \int_{\partial\Omega_{L} \setminus (\partial\Omega_{L} \cap \partial\Omega)} w_{L}^{n+1} u_{G}^{n} d\Gamma + \eta \int_{\partial\Omega_{L} \cap \Gamma_{c}} \bar{u}^{n+1} w_{L}^{n+1} d\Gamma
$$
Global-Local Enrichments for Heat Equation

• Defining Step: Global space is enriched with local solutions

\[ \phi^{g1,n+1}_\alpha(x) = \varphi_\alpha(x) u^{n+1}_L(x) \]

Find \( u^{n+1}_G(x) \in S^{GFEM,n+1}_G(\Omega) = S^{FEM}_G + \{ \varphi_\alpha u^{g1,n+1}_\alpha, \alpha \in I^{g1} \} \)

where \( u^{g1,n+1}_\alpha(x) = u_\alpha u^{n+1}_L(x) \in \chi^{n+1}_\alpha, u_\alpha \in \mathbb{R} \)

• Discretization spaces updated on-the-fly with global-local enrichment functions
Time- or load-dependent GFEM space

- Updating local solutions at each step leads to time- or load-dependency of global-local enrichments and approximation spaces:

\[ \phi^{gl,n+1}_\alpha(x) = \varphi_\alpha(x) u^{n+1}_L(x) \]

- An issue in:
  - **Transient problems**: How to formulate time integration scheme? (O’Hara et al. 2010)
  - **Nonlinear problems**: How to start Newton-Raphson iteration when solution space changes? Solution vector at load step (n) cannot be used with shape functions at load step (n+1)
  - This is also an issue in
    - analytically defined enrichment functions if they are added/deleted between time/load steps
    - adaptive FEMs
Time-dependent GFEM Spaces: Elasto-plastic materials

- Nonlinear solution based on incremental load steps:
  \[ \mathbf{u}_{n+1} = \mathbf{u}_n + \Delta \mathbf{u}_{n+1} \]
  \[ \sigma(\mathbf{u}_{n+1}) = \sigma(\mathbf{u}_n + \Delta \mathbf{u}_{n+1}) \]

  Shape functions at previous time step
  Shape functions at current time step

- Elasto-plastic behavior:
  \[ \sigma(\mathbf{u}_{n+1}) = C : \varepsilon^m_n + \sigma(\Delta \mathbf{u}_{n+1}) \]

  Total stress is **linear** in previous converged solution

![Graph showing stress-strain relationship](image)
Time-dependent GFEM Spaces: Elasto-plastic materials

- Solution vector at load step (n) cannot be used with shape functions at load step (n+1)
- Solve a linear elastic “predictor” problem to get the total solution at load step (n+1) using shape functions for step (n+1)

\[
\int_{\Omega^u} \varepsilon(u^{(1)}_{n+1}) : C : \varepsilon(\delta u) \, d\Omega + \eta \int_{\Gamma^u} u^{(1)}_{n+1} \cdot \delta u \, d\Gamma = \int_{\Gamma^t} t_{n+1} \cdot \delta u \, d\Gamma \\
+ \eta \int_{\Gamma^u} \bar{u}_{n+1} \cdot \delta u \, d\Gamma + \int_{\Omega^u} \left( \varepsilon_n^{(p)} + \varepsilon_n^{(\theta)} \right) : C : \varepsilon(\delta u) \, d\Omega
\]

- Discretize using Current shape functions: n+1 step
- RHS uses: Current external loads and thermal strains, previous plastic strains

\[
u^{(1)}_{n+1} = \bar{N}^u_{n+1} d^{u(1)}_{n+1} \\
\varepsilon^{(1)}_{n+1} = \bar{B}^u_{n+1} d^{u(1)}_{n+1}
\]
Time-dependent GFEM Spaces: Elasto-plastic materials

- Solve a *linear elastic* “predictor” problem to get the total solution at load step (n+1) using shape functions for step (n+1)

\[
K_{elas,n+1}^{u} d_{n+1}^{u(1)} = f_{ext,n+1}^{u} + f_{int,n+1}^{u(0)}
\]

- Yields predictor for **total solution** at initial Newton iteration
- Solution provides initial guess for Newton-Raphson at step (n+1)
- No interpolation of quantities between meshes like in adaptive FEM
- *All information available at integration points which are NOT time-dependent*
Numerical example: Laser-heated beam

- Coupon beam subjected to transient Gaussian laser heating

- **hp-GFEM** (locally refined)
  - Direct FE Analysis (DFEA)

- **GFEM$^{gl}$** (coarse, structural-scale)
  - Has special enrichment functions generated from local problem solutions

- **GFEM$^{gl}$** ($hp$-adapted local problem)
Numerical example: Laser-heated beam

- **Case 1:** Stationary sharp heating, then cooling to room temperature
Numerical example: Laser-heated beam

- **Case 1:** Stationary sharp heating, then cooling to room temperature

GFEM\(^{gl}\) captures localized temperature gradients, stresses, and residual deformations on a coarse-scale, uniform mesh.
Numerical example: Laser-heated beam

- $\text{GFEM}^\text{gl}$ vs. direct ($hp$-GFEM) analysis:
  - Pointwise quantities at maximum load/temperature:

  ![Graphs showing temperature and Von Mises stress vs. position.]
Numerical example: Laser-heated beam

- **Case 2**: Moving sharp flux

![Diagram showing GFEM and hp-GFEM meshes with temperature and axial stress comparison]
Numerical example: Laser-heated beam

- GFEM\textsuperscript{gl} vs. \textit{hp}-GFEM solutions in time: Axial stress

\begin{itemize}
  \item \( t = 0.25 \)
  \item \( t = 3.0 \)
  \item \( t = 5.0 \)
  \item \( t = 8.0 \)
\end{itemize}
GFEM$^g_l$ Solution of a Hat-Stiffened Panel

- Stiffened panel with 168 spot welds*

Represent spot welds using global-local enrichments
- Use a coarse mesh at global scale

*Panel geometry and properties courtesy of Air Force Research Laboratory, OH, USA

Ti-6242S properties:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>17100</td>
<td>ksi</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.325</td>
<td>–</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>152.0</td>
<td>ksi</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$4.28 \times 10^{-6}$</td>
<td>°F$^{-1}$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.8755</td>
<td>$\frac{\text{ft-lbf}}{\text{s-in} \cdot \degree \text{F}}$</td>
</tr>
<tr>
<td>$\rho c$</td>
<td>14.04</td>
<td>$\frac{\text{ft-lbf}}{\text{in}^3 \cdot \degree \text{F}}$</td>
</tr>
</tbody>
</table>
Case 1: Mechanical load only: Uniform pressure on skin panel

- Linear elastic response
- Use symmetry properties to reduce problem size
- 44 spot welds
GFEM$^\text{gl}$ Solution of a Hat-Stiffened Panel

- Global mesh with hexahedron elements: Spot welds are *not* discretized at this scale

Dirichlet (blue) and Neumann (red) BCs
GFEM$_{gl}$ Solution of a Hat-Stiffened Panel

- Global problem provides BCs for local problems
- Define and solve in **parallel**, a local problem for each spot weld
- Use local solutions as enrichments in global mesh (red nodes)
GFEM$^g_l$ Solution of a Hat-Stiffened Panel

- GFEM$^g_l$ results: Deformed configuration and von Mises stress

**Enriched** global problem, deformed shape and Von Mises stress
GFEM$^g_l$ Solution of a Hat-Stiffened Panel

- GFEM$^g_l$ results: Von Mises stress

Localized stress fields are well captured on a coarse global (HEX8) mesh

Von Mises Stress on the skin panel

Von Mises Stress on the stiffeners
Case 2: Transient nonlinear thermo-mechanical analysis

Parallel analysis of spot weld connections to predict stresses/plastic deformation
Case 2: Boundary conditions

- Steady, uniform pressure + heat flux
  - $\bar{p}_0 = 2.0$ psi
  - $\bar{f}_0 = 5.0 \text{ ft-lbf/s-in}^2$

- Fixed against in-plane deformation
  - $\bar{u}_x = 0$
  - $\bar{u}_z = 0$

- Specified temperature along all edges
  - $\bar{\theta} = 70^\circ F$

- Springs
**Goal:** Find critical location of a localized thermal load

- *Localized, transient heat flux* in Regions A and C:

\[
\bar{f}(x, t) = \frac{I_0}{2\pi a^2}g(t)h(z)\exp\left(-\frac{(x - b)^2}{2a^2}\right) + \bar{f}_0
\]

Temporal variation of peak (Gaussian) flux:
GFEM\textsuperscript{gl} Solution of a Hat-Stiffened Panel

- GFEM\textsuperscript{gl} global + local meshes
- Same global mesh as before

Coarse, global hexahedral mesh (capture structural geometry)

Typical tetrahedral weld mesh (capture local stress concentrations/plasticity in welds)

Tetrahedral local region mesh (capture sharp heat flux)
GFEM\textsuperscript{gl} Solution of a Hat-Stiffened Panel

- Panel/stiffeners connected by series of 44 spot welds

GFEM\textsuperscript{gl} global + local problem sizes

<table>
<thead>
<tr>
<th>Problem size (dofs)</th>
<th>Heat transfer</th>
<th>Thermoplasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial global</td>
<td>27,888</td>
<td>209,160</td>
</tr>
<tr>
<td>Enriched global (A)</td>
<td>28,480</td>
<td>210,936</td>
</tr>
<tr>
<td>Enriched global (C)</td>
<td>28,436</td>
<td>210,804</td>
</tr>
<tr>
<td>Local Full spot weld</td>
<td>7,966</td>
<td>95,592</td>
</tr>
<tr>
<td>Half-spot weld</td>
<td>4,179</td>
<td>50,148</td>
</tr>
<tr>
<td>Region A</td>
<td>5,807</td>
<td>69,684</td>
</tr>
<tr>
<td>Region C</td>
<td>5,965</td>
<td>71,580</td>
</tr>
</tbody>
</table>

Only \(~1,000\) extra global dofs from global–local enrichments (< 1% increase)

Estimated \(hp\)-GFEM/\(hp\)-FEM (direct analysis) problem size \(\approx 4.5\) million dofs
GFEM$^{gl}$ Solution of a Hat-Stiffened Panel

- **Region A**

  Von Mises stress

  Temperature

  Movie
GFEM$^g$ Solution of a Hat-Stiffened Panel

- **Region A**
  - Von Mises stress (panel + stiffeners)

Local thermal stress effects concentrated in panel

Movie
GFEM$^g_l$ Solution of a Hat-Stiffened Panel

- Region C

Von Mises stress

Temperature

Movie
**GFEM\textsuperscript{gl} Solution of a Hat-Stiffened Panel**

- **Region C**
  - Von Mises stress (panel + stiffeners)

Local thermal stress effects in panel shielded by proximity to **stiffeners**

![Movie](image-url)
Region C
- $t = 2.0s$ (maximum thermal load)

Temperature

Localized spot weld stresses in vicinity of sharp heating

von Mises stress
**GFEM\textsuperscript{gl} Solution of a Hat-Stiffened Panel**

- **GFEM\textsuperscript{gl} parallel performance**
  - Single time/load step—all solution phases (local + enriched global) considered
  - Up to 24 CPUs

- Good speedup on small number of threads
- Efficiency deteriorates as number of CPUs increases (expected)

---

Number of local problems \(\approx\) number of threads—difficult to achieve good load balance
GFEM\textsuperscript{gl} Solution of a Hat-Stiffened Panel

- GFEM\textsuperscript{gl} parallel performance
  - Time spent in each solution phase vs. number of parallel threads:

![Graph showing time spent in each solution phase vs. number of threads.]

Total solution wall time

Fraction of total solution time

Wall times spent in enriched/local problems scale \( \sim \) uniformly—no bottlenecks!

- Enriched global assembly/solution
- Local assembly/solution
Summary

- GFEM$^g_l$ for large, nonlinear, coupled thermo-structural problems exhibiting phenomena spanning multiple spatial scales of interest

- Time-dependent global–local enrichments for capturing nonlinear (elasto-plastic) effects at disparate structural scales

- Fine-scale problems parallelizable; efficiently resolve localized plasticity *at the fine scale*, maintain coarse, global structural mesh
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caduarte@illinois.edu

http://gfem.cee.illinois.edu/