UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

Generalized finite element approaches for analysis of localized non-linear thermo-mechanical effects

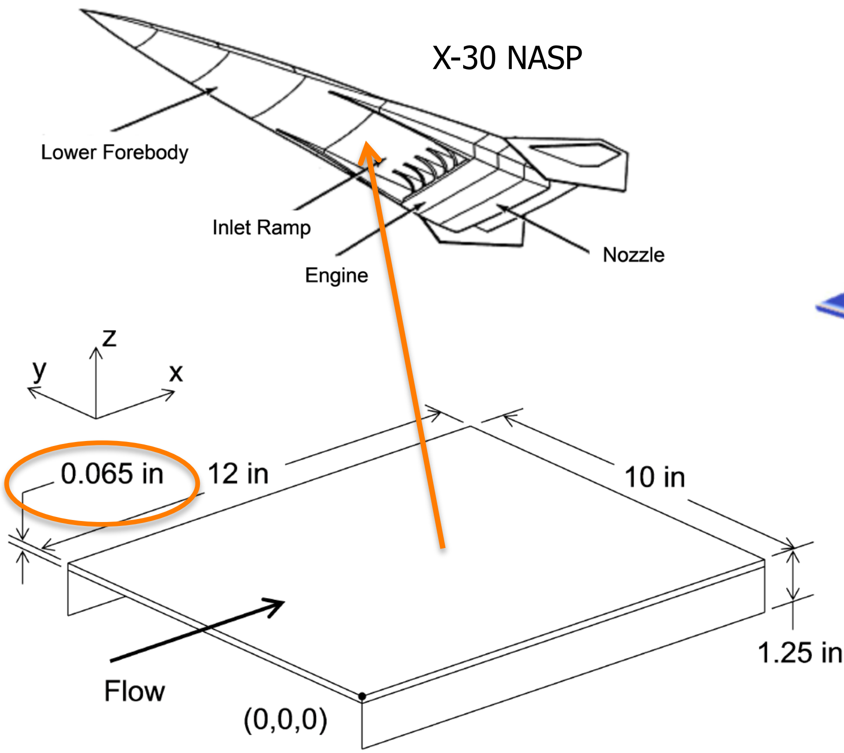
To the memory of Prof. Ted Belytschko

C. Armando Duarte and Julia Plews
Dept. of Civil and Environmental Engineering
Computational Science and Engineering

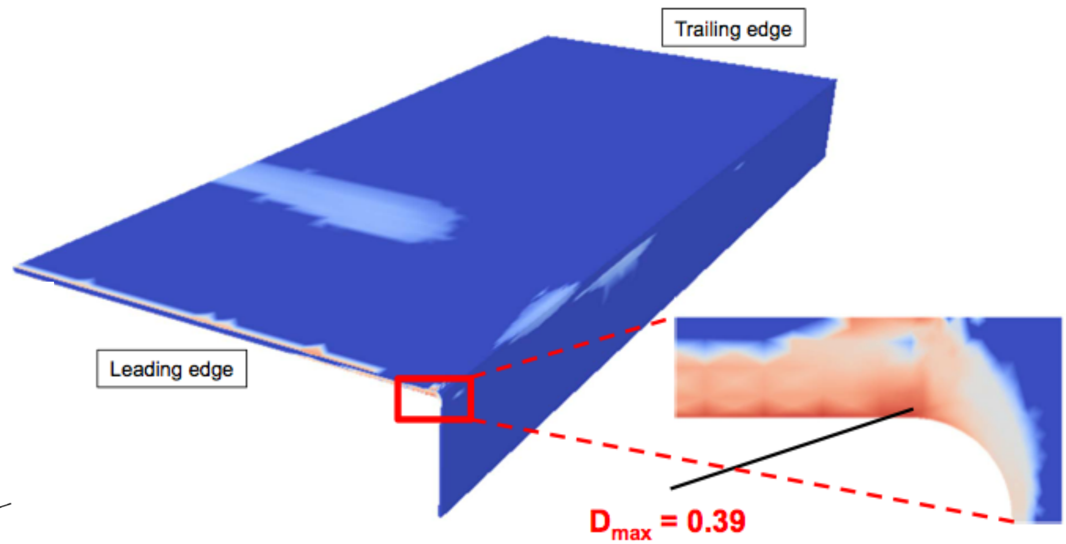
Presented at the 8th International Workshop on Meshfree Methods for PDEs
Bonn, Germany, September, 2015

Motivation: Multiscale Structural Analysis

- Thermo, mechanical and acoustic loads on hypersonic aircrafts lead to highly localized non-linear stress fields: 3-D finite element models with fine meshes are required



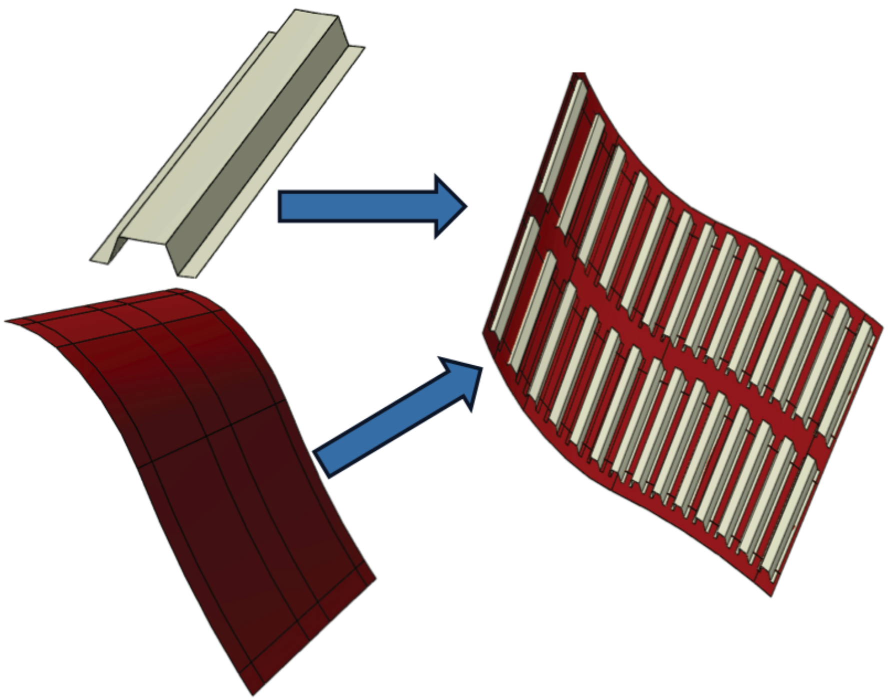
Representative hypersonic skin panel
[Sobotka et al., 2013]

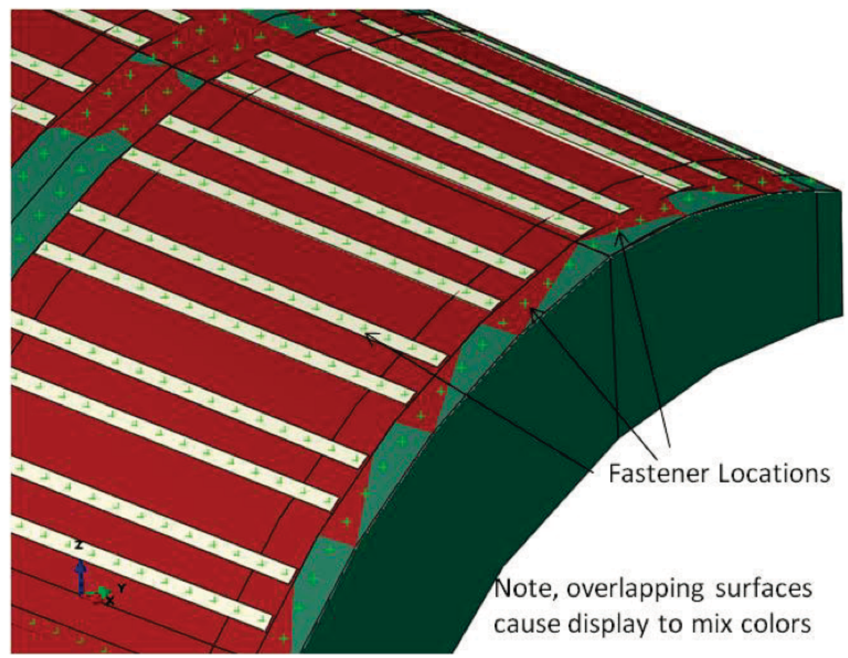


- Highly localized non-linear 3-D effects
- Most of the structure remains linear elastic
- 3-D FEM: Large aspect ratio of elements may lead to numerical instabilities

Motivation: Multiscale Structural Analysis

- Hypersonic aircraft panels are assembled from sub-components using hundreds of fasteners or spot welds.
- Multiple spatial scales: Skin panel, stiffeners, spot welds.

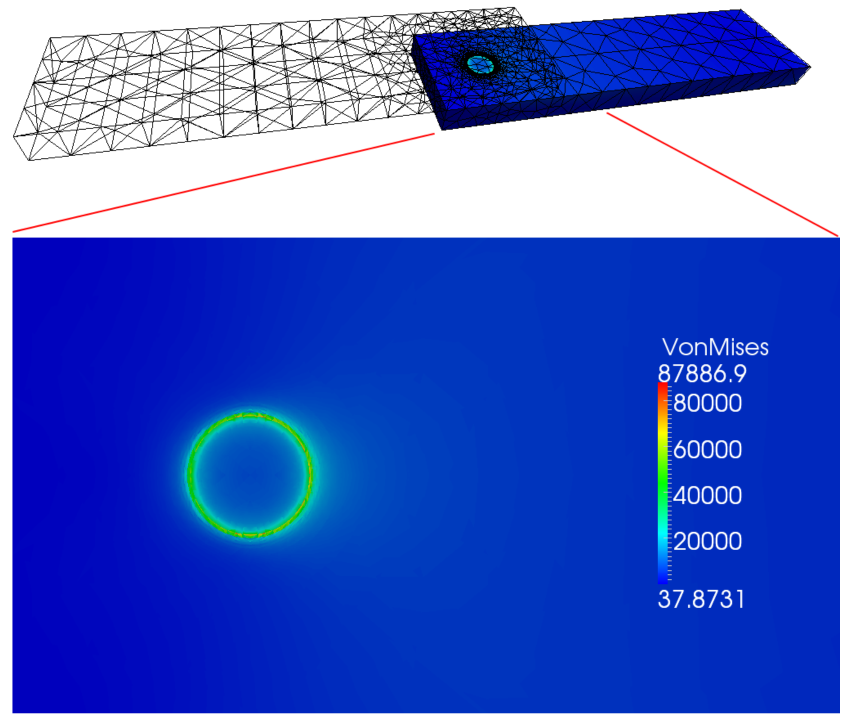




Panel 1 to Sub-Structure Attachment
[AFRL-RB-WP-TR-2012-0280]

Motivation: Multiscale Structural Analysis

- Representation of a spot weld in the FEM requires detailed meshing.
- Hundreds of spot welds in one single panel: Not feasible to mesh them all.
- Multi-point constraint is used instead in the industry: This leads to *mesh dependent solutions even far from spot welds!*
- **Strategy:**
- Formulate a two-scale GFEM for this class of problems;
- Keep global mesh *coarse* and resolve spot welds through enrichment functions computed in parallel.



3-D Adaptive FEM mesh and von Mises stress in a lap joint with a spot weld

Outline

- Motivation
- Generalized finite element methods: Basic ideas
- Bridging scales with GFEM:
  - Global-local enrichments for heat equation and nonlinear thermo-mechanical problems
- Numerical examples
- Conclusions

Generalized Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

$$\mathbb{S}_{GFEM} = \mathbb{S}_{FEM} + \mathbb{S}_{ENR}$$

Low order FEM space

Enrichment space with functions related to the given problem

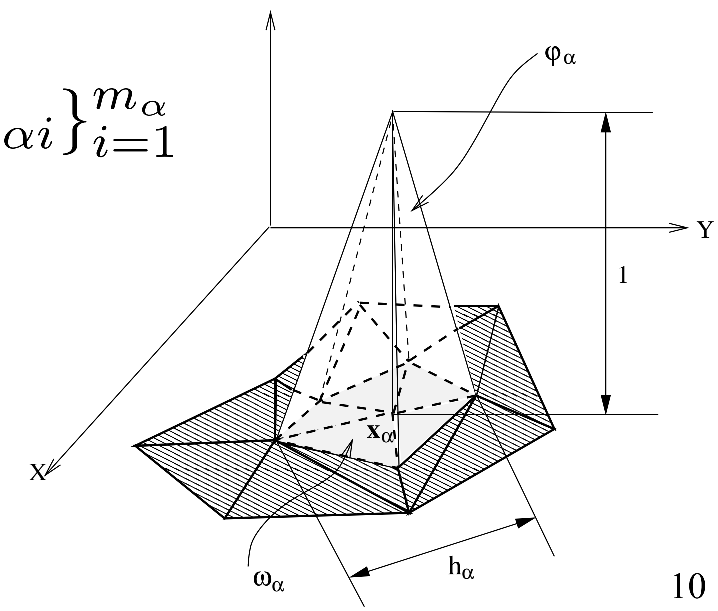
$$\mathbb{S}_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R}$$

$$\mathbb{S}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \mathrm{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}$$

$$L_{\alpha i} \in \chi_\alpha(\omega_\alpha)$$

Enrichment function

Patch space



Generalized Finite Element Method

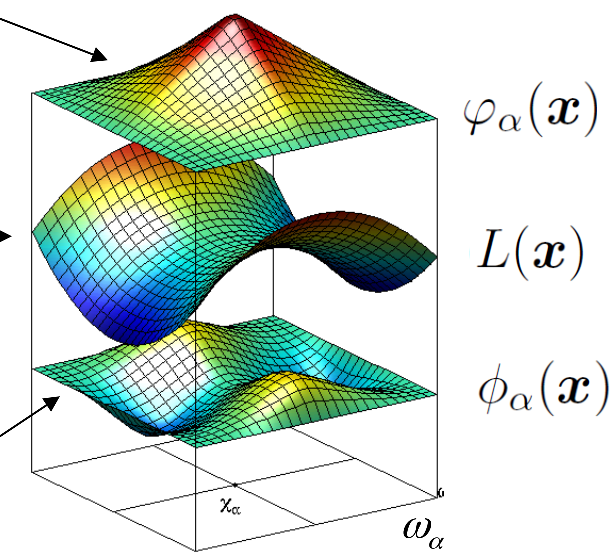
$$\mathbb{S}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}$$

$$\phi_{\alpha i}(x) = \varphi_\alpha(x) L_{\alpha i}(x) \qquad \sum_\alpha \varphi_\alpha(\boldsymbol{x}) = 1$$

Linear FE shape function

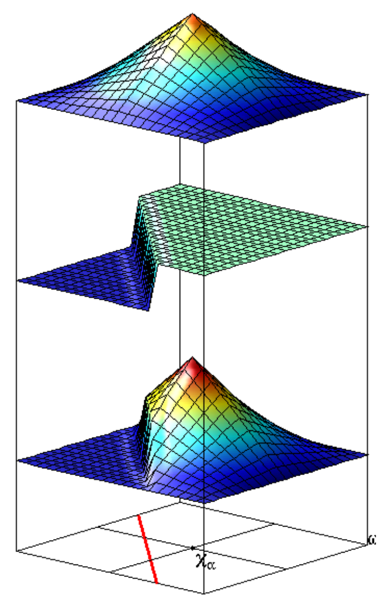
Enrichment function

GFEM shape function



[Oden, Duarte & Zienkiewicz, 1996]

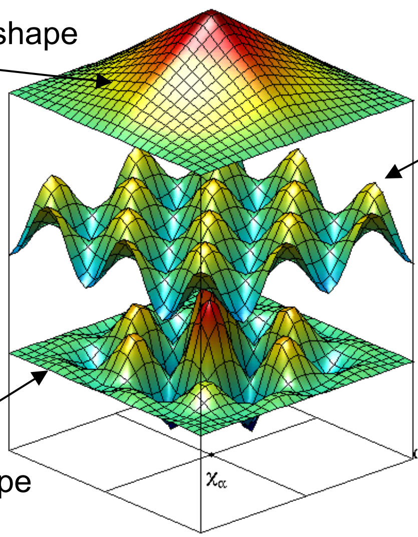
• Allows construction of shape functions incorporating a-priori knowledge about solution



Discontinuous enrichment [Moes et al., 1999]

Bridging Scales with Global-Local Enrichment Functions*

- Enrichment functions computed from solution of local boundary value problems: <u>Global-Local enrichment functions</u>



Enrichment = Numerical solutions of BVP

- **Idea:** *Use available numerical solution at a simulation step to build shape functions for next step* (quasi-static, transient, non-linear, etc.)
- Enrichment functions are produced numerically on-the-fly through a global-local analysis
- Use a *coarse* mesh enriched with Global-Local (GL) functions
- $GFEM^{gl}$ = GFEM with global-local enrichments

*[Duarte et al. 2005]

Global-Local Enrichments for Heat Equation

$$\rho c \frac{\partial u}{\partial t} = \nabla(\kappa(\boldsymbol{x})\nabla u) + Q(\boldsymbol{x}, t) \quad \text{in} \quad \Omega$$

where $u(\boldsymbol{x}, t)$ is the temperature field, ρc is the volumetric heat capacity and $Q(\boldsymbol{x}, t)$ is the internal heat source. $\kappa(\boldsymbol{x})$ may be oscillatory.

$$-\kappa \frac{\partial u}{\partial n} = \eta\left(\bar{u} - u\right) \quad \text{on} \quad \Gamma_c$$

$$-\kappa \frac{\partial u}{\partial n} = \bar{f} \quad \text{on} \quad \Gamma_f$$

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}) \quad \text{at} \quad t^0$$

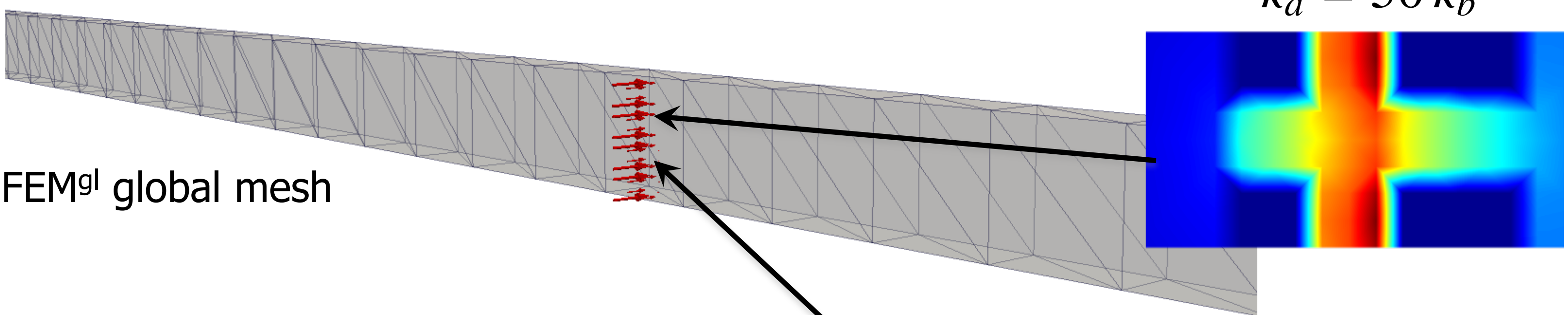
where $u^0(\boldsymbol{x})$ is the prescribed temperature field at time $t = t^0$

Domain Subjected to Sharp Laser Flux

- **Goal**: Solve with GFEM^{gl} on the mesh shown below

Local material heterogeneity:

$\kappa_a = 50\,\kappa_b$



GFEM^{gl} global mesh

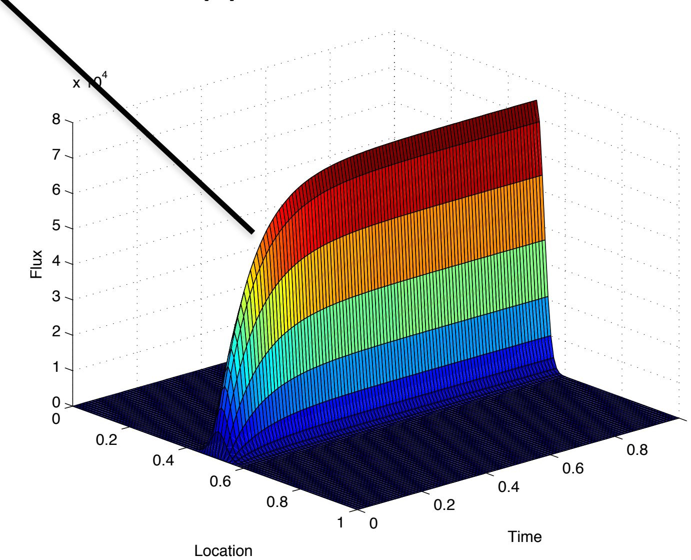
Sharp (Gaussian), localized heat flux applied as shown

Laser flux:

$$\bar{f}(\boldsymbol{x},t) = I_0 * f(t) * \frac{1}{2\pi a^2} * G(\boldsymbol{x},b,a)$$

$$f(t) = 1 - \exp(-\gamma * t)$$

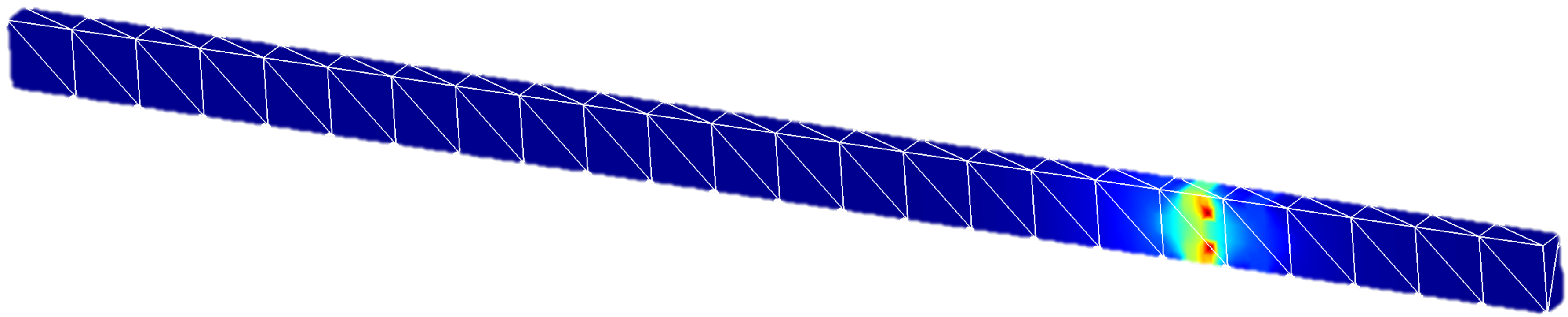
$$G(\boldsymbol{x},b,a) = \exp\left(\frac{-(x-b)^2}{2a^2}\right)$$



Convection BCs applied everywhere else

Global-Local Enrichments for Heat Equation

Let $u_G^n(\boldsymbol{x}) \in \mathbb{S}_G^{GFEM,n}(\Omega)$ be the GFEM solution at time $t = t^n = n\Delta t$



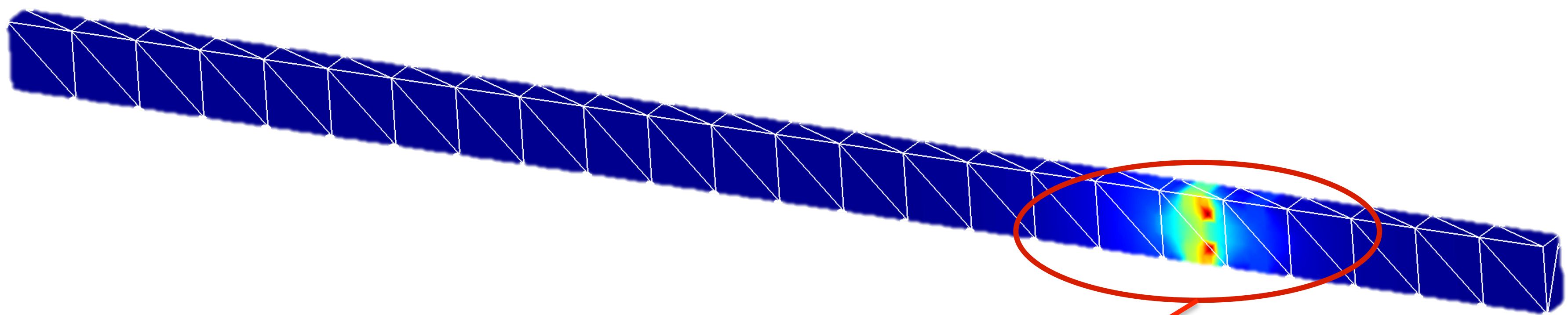
Find $u_G^n \in \mathbb{S}_G^{\mathrm{GFEM},n}(\Omega_G)$ such that, $\forall\ w_G^n \in \mathbb{S}_G^{\mathrm{GFEM},n}(\Omega_G)$

$$\frac{\rho c}{\Delta t}\int_\Omega w_G^n u_G^n d\Omega + \int_\Omega \left(\nabla w_G^n\right)^T \kappa \nabla u_G^n d\Omega + \eta \int_{\Gamma_c} w_G^n u_G^n d\Gamma =$$

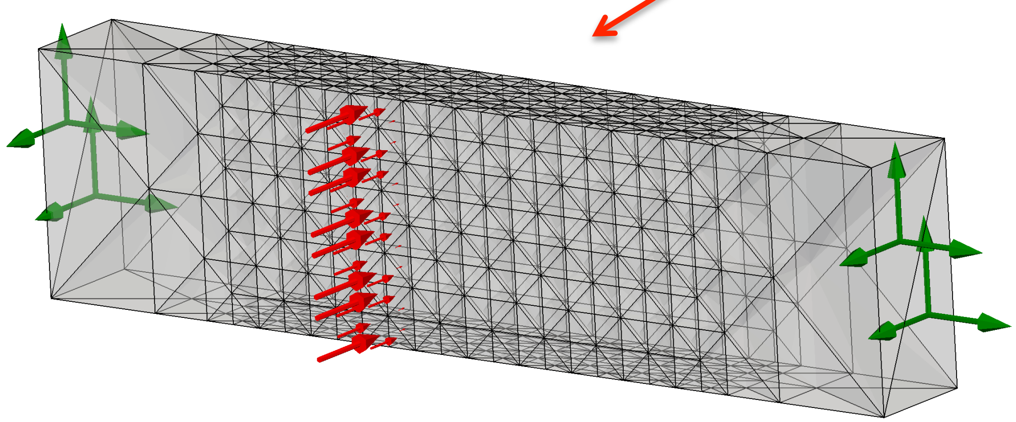
$$\frac{\rho c}{\Delta t}\int_\Omega w_G^n u_G^{n-1} d\Omega + \int_{\Gamma_f} \bar{f}^n w_G^n d\Gamma + \eta \int_{\Gamma_c} \bar{u}^n w_G^n d\Gamma + \int_\Omega Q^n w_G^n d\Omega$$

Global-Local Enrichments for Heat Equation

Let $u_G^n(\boldsymbol{x}) \in \mathbb{S}_G^{GFEM,n}(\Omega)$ be the GFEM solution at time $t = t^n = n\Delta t$

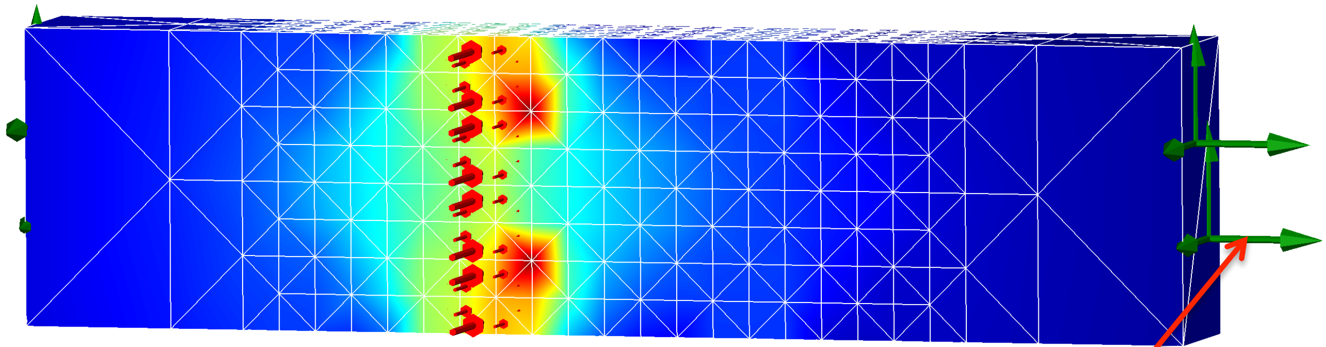


- Define local domain around the laser flux location at time $t = t^{n+1}$



Global-Local Enrichments for Heat Equation

- Solve following *local problem* at time $t = t^{n+1}$ using, e.g., *hp*-GFEM



$$\text{Find } u_L^{n+1} \in \mathbb{S}_L^{\text{GFEM},n+1}(\Omega_L) \text{ such that, } \forall\ w_L^{n+1} \in \mathbb{S}_L^{\text{GFEM},n+1}(\Omega_L)$$

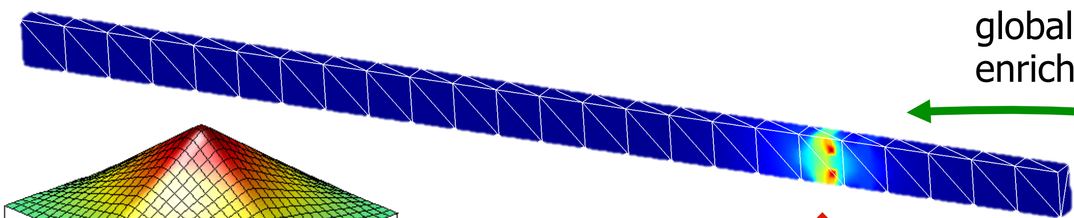
$$\int_{\Omega_L} \left(\nabla w_L^{n+1}\right)^T \kappa \nabla u_L^{n+1} d\Omega + \eta \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \Gamma_f)} w_L^{n+1} u_L^{n+1} d\Gamma$$

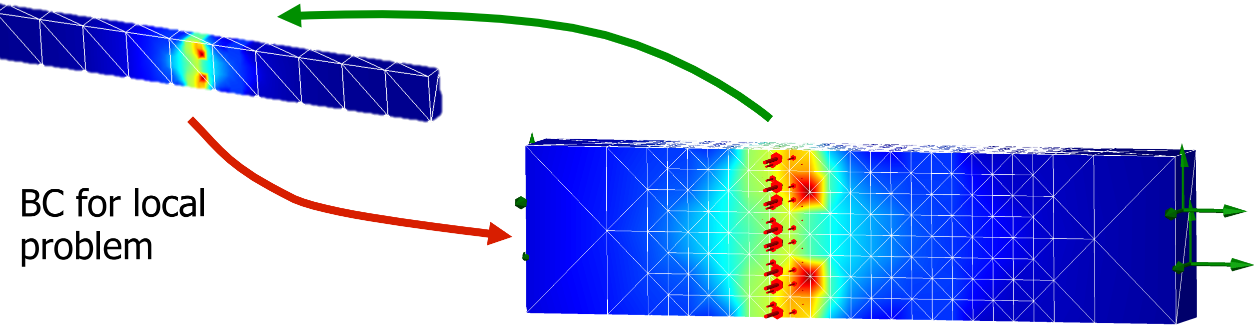
$$= \int_{\Omega_L} Q^{n+1} w_L^{n+1} d\Omega + \int_{\partial\Omega_L \cap \Gamma_f} \bar{f}^{n+1} w_L^{n+1} d\Gamma$$

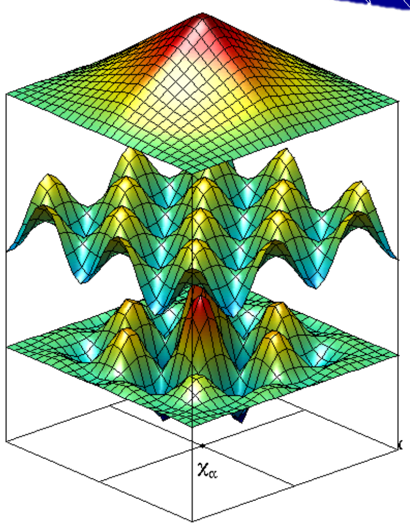
$$+ \eta \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega)} w_L^{n+1} u_G^n d\Gamma + \eta \int_{\partial\Omega_L \cap \Gamma_c} \bar{u}^{n+1} w_L^{n+1} d\Gamma$$

Global-Local Enrichments for Heat Equation

- **Defining Step:** Global space is enriched with local solutions







$$\phi_\alpha^{\mathrm{gl},n+1}(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})\, u_L^{n+1}(\boldsymbol{x})$$

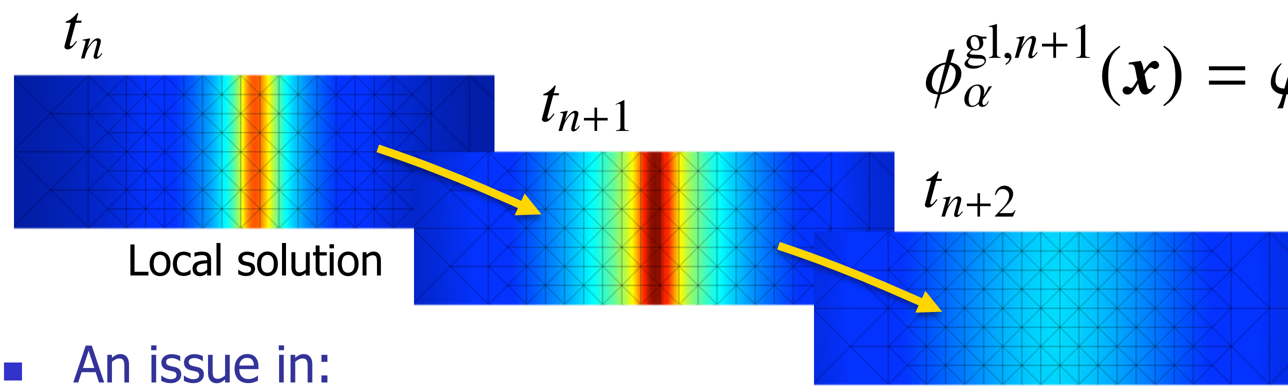
$$\text{Find } u_G^{n+1}(\boldsymbol{x}) \in \mathbb{S}_G^{\mathrm{GFEM},n+1}(\Omega) = \mathbb{S}_G^{\mathrm{FEM}} + \left\{\varphi_\alpha u_\alpha^{\mathrm{gl},n+1},\ \alpha \in \mathcal{I}^{\mathrm{gl}}\right\}$$

$$\text{where } u_\alpha^{\mathrm{gl},n+1}(\boldsymbol{x}) = \underline{u}_\alpha u_L^{n+1}(\boldsymbol{x}) \in \chi_\alpha^{n+1},\ \underline{u}_\alpha \in \mathbb{R}$$

- Discretization spaces updated on-the-fly with global-local enrichment functions

Time- or load-dependent GFEM space

- Updating local solutions at each step leads to time- or load-dependency of global-local enrichments and approximation spaces:



$$\phi_\alpha^{\mathrm{gl},n+1}(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})\, u_L^{n+1}(\boldsymbol{x})$$

- An issue in:
  - Transient problems: How to formulate time integration scheme? (O'Hara et al. 2010)
  - Nonlinear problems: How to start Newton-Raphson iteration when solution space changes? Solution vector at load step (n) cannot be used with shape functions at load step (n+1)
  - This is also an issue in
    - analytically defined enrichment functions if they are added/deleted between time/load steps
    - adaptive FEMs

Time-dependent GFEM Spaces: Elasto-plastic materials

- Nonlinear solution based on incremental load steps:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta\mathbf{u}_{n+1} \qquad \boldsymbol{\sigma}(\mathbf{u}_{n+1}) = \boldsymbol{\sigma}(\mathbf{u}_n + \Delta\mathbf{u}_{n+1})$$

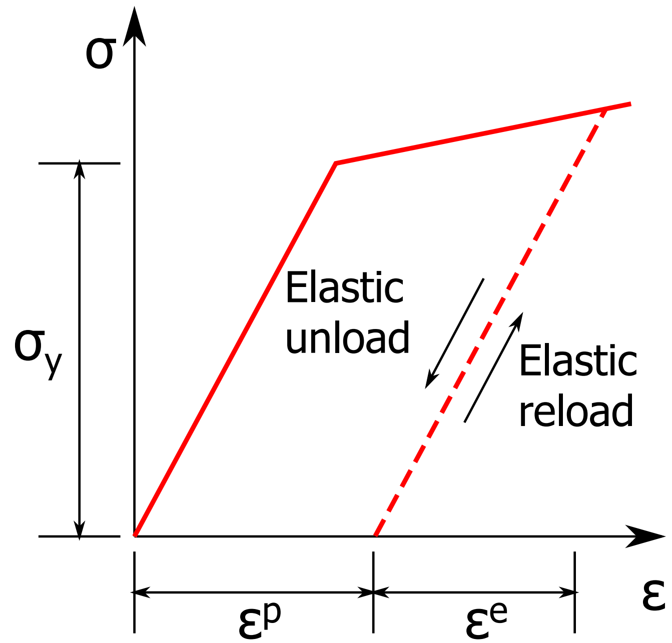
Shape functions at previous time step

Shape functions at current time step

- Elasto-plastic behavior:

$$\boldsymbol{\sigma}(\mathbf{u}_{n+1}) = \mathbf{C} : \boldsymbol{\varepsilon}_n^m + \boldsymbol{\sigma}(\Delta\mathbf{u}_{n+1})$$

Total stress is **linear** in previous converged solution



Time-dependent GFEM Spaces: Elasto-plastic materials

- Solution vector at load step (n) cannot be used with shape functions at load step (n+1)
- Solve a *linear elastic* "predictor" problem to get the total solution at load step (n+1) using shape functions for step (n+1)

$$\int_{\Omega^{\mathbf{u}}} \boldsymbol{\varepsilon}(\mathbf{u}_{n+1}^{(1)}) : \mathbf{C} : \boldsymbol{\varepsilon}(\delta \mathbf{u}) \, \mathrm{d}\Omega + \eta \int_{\Gamma^{\mathbf{u}}} \mathbf{u}_{n+1}^{(1)} \cdot \delta \mathbf{u} \, \mathrm{d}\Gamma = \int_{\Gamma^{\mathbf{t}}} \bar{\mathbf{t}}_{n+1} \cdot \delta \mathbf{u} \, \mathrm{d}\Gamma$$

$$+ \eta \int_{\Gamma^{\mathbf{u}}} \bar{\mathbf{u}}_{n+1} \cdot \delta \mathbf{u} \, \mathrm{d}\Gamma + \int_{\Omega^{\mathbf{u}}} \left(\boldsymbol{\varepsilon}_n^p + \boldsymbol{\varepsilon}_{n+1}^\theta \right) : \mathbf{C} : \boldsymbol{\varepsilon}(\delta \mathbf{u}) \, \mathrm{d}\Omega$$

- Discretize using *Current* shape functions: n+1 step
- RHS uses: *Current* external loads and thermal strains, *previous* plastic strains

$$\mathbf{u}_{n+1}^{(1)} = \bar{\mathbf{N}}_{n+1}^{\mathbf{u}} \mathbf{d}_{n+1}^{\mathbf{u}(1)}$$

$$\boldsymbol{\varepsilon}_{n+1}^{(1)} = \bar{\mathbf{B}}_{n+1}^{\mathbf{u}} \mathbf{d}_{n+1}^{\mathbf{u}(1)}$$

Time-dependent GFEM Spaces: Elasto-plastic materials

- Solve a *linear elastic* "predictor" problem to get the total solution at load step (n+1) using shape functions for step (n+1)

$$\mathbf{K}^{\mathbf{u}}_{\text{elas},n+1}\mathbf{d}^{\mathbf{u}(1)}_{n+1} = \mathbf{f}^{\mathbf{u}}_{\text{ext},n+1} - \mathbf{f}^{\mathbf{u}(0)}_{\text{int},n+1}$$

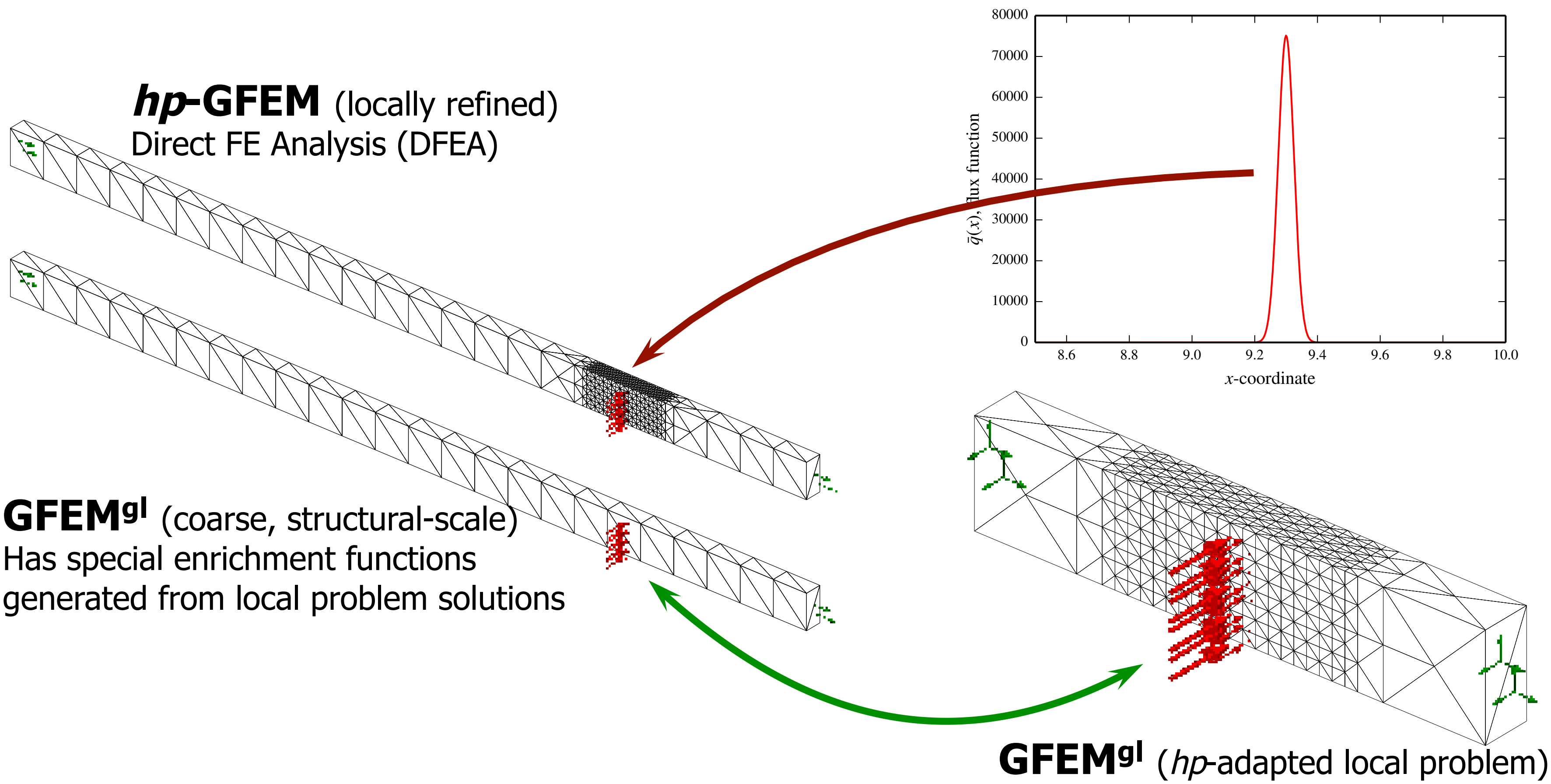
Current shape functions and external load

Current shape fns; *current* thermal, *previous* plastic strains

- Yields predictor for total solution at initial Newton iteration
- Solution provides initial guess for Newton-Raphson at step (n+1)
- No interpolation of quantities between meshes like in adaptive FEM
- *All information available at integration points which are NOT time-dependent*

Numerical example: Laser-heated beam

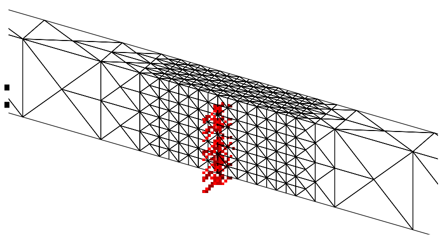
- Coupon beam subjected to transient Gaussian laser heating



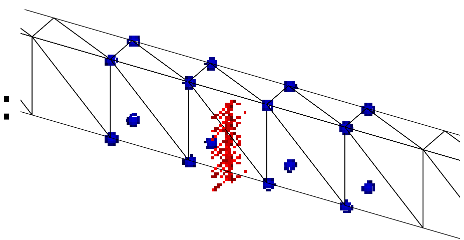
Numerical example: Laser-heated beam

- **Case 1:** Stationary sharp heating, then cooling to room temperature

hp-GFEM mesh:



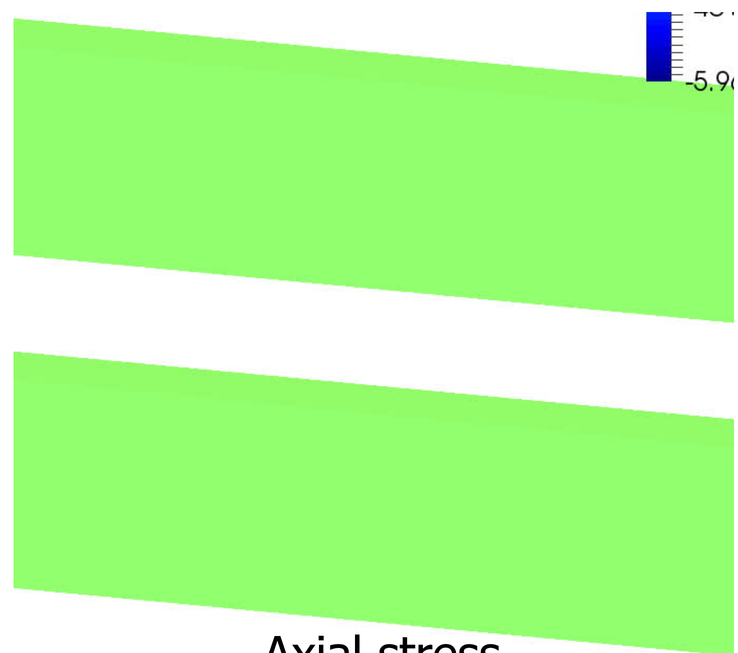
GFEM^{gl} mesh:



GFEM^{gl}

hp-GFEM

Temperature field

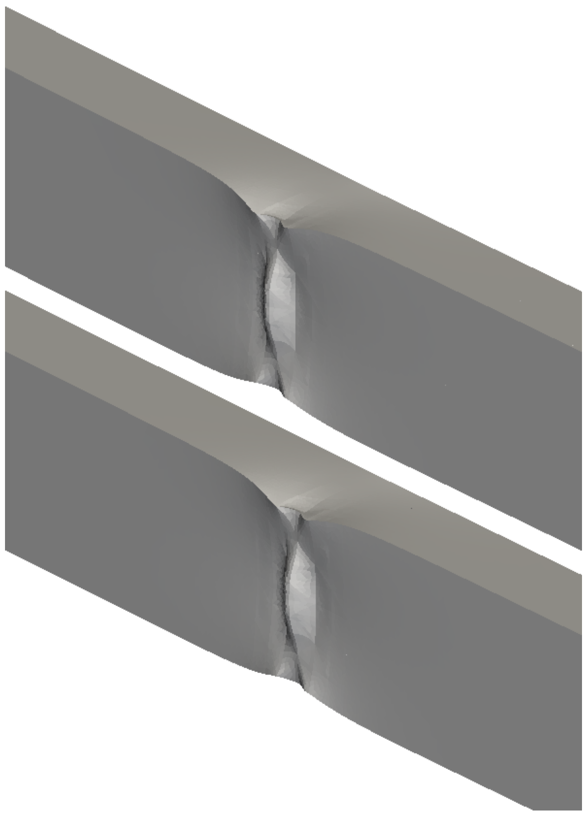


Axial stress

Numerical example: Laser-heated beam

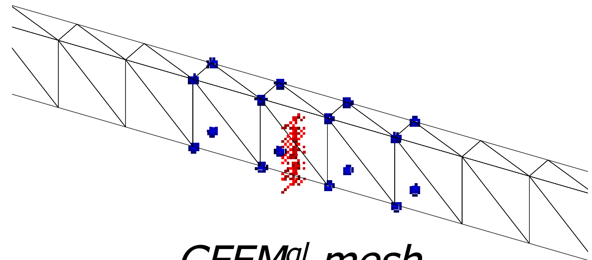
- **Case 1:** Stationary sharp heating, then cooling to room temperature

GFEM^{gl}



hp-GFEM

Residual deformation, fully cooled

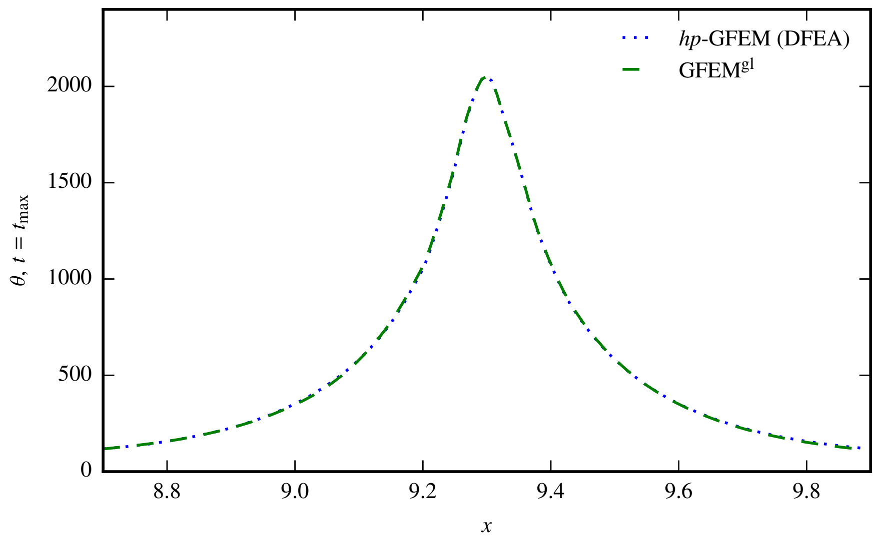


GFEM^{gl} mesh

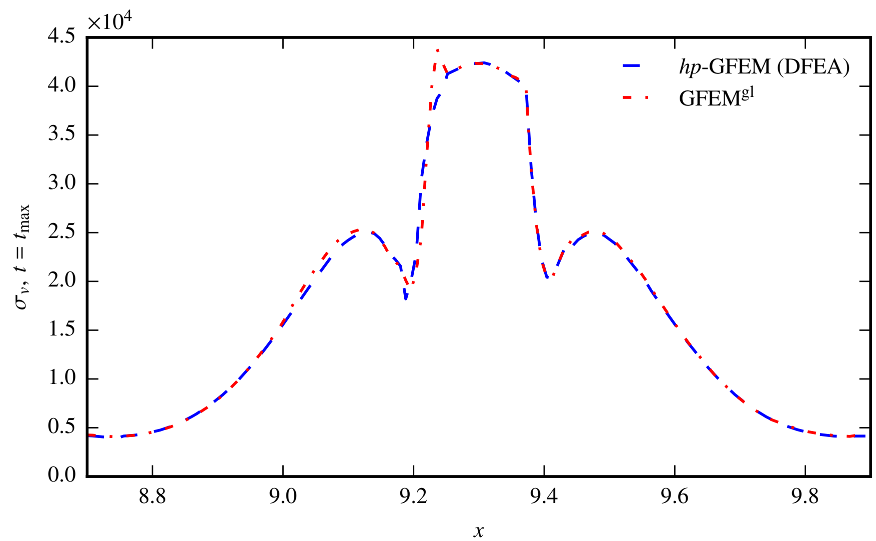
GFEM^{gl} captures localized temperature gradients, stresses, and residual deformations on a **coarse-scale, uniform mesh**.

Numerical example: Laser-heated beam

- GFEM^{gl} vs. direct (*hp*-GFEM) analysis:
  - Pointwise quantities at maximum load/temperature:



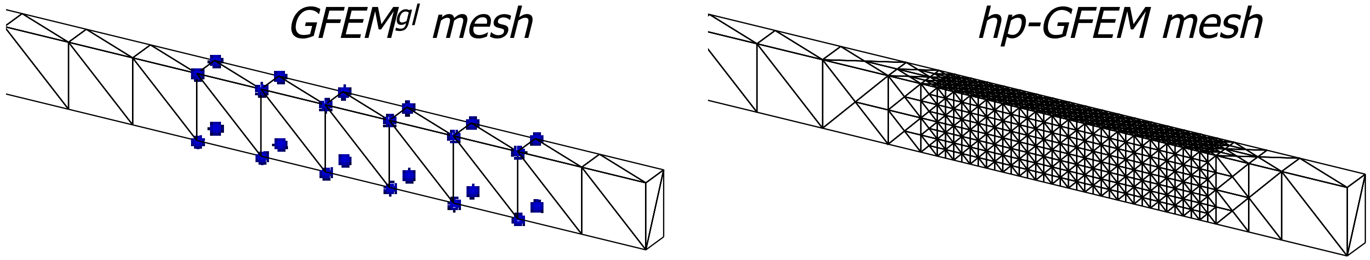
Temperature

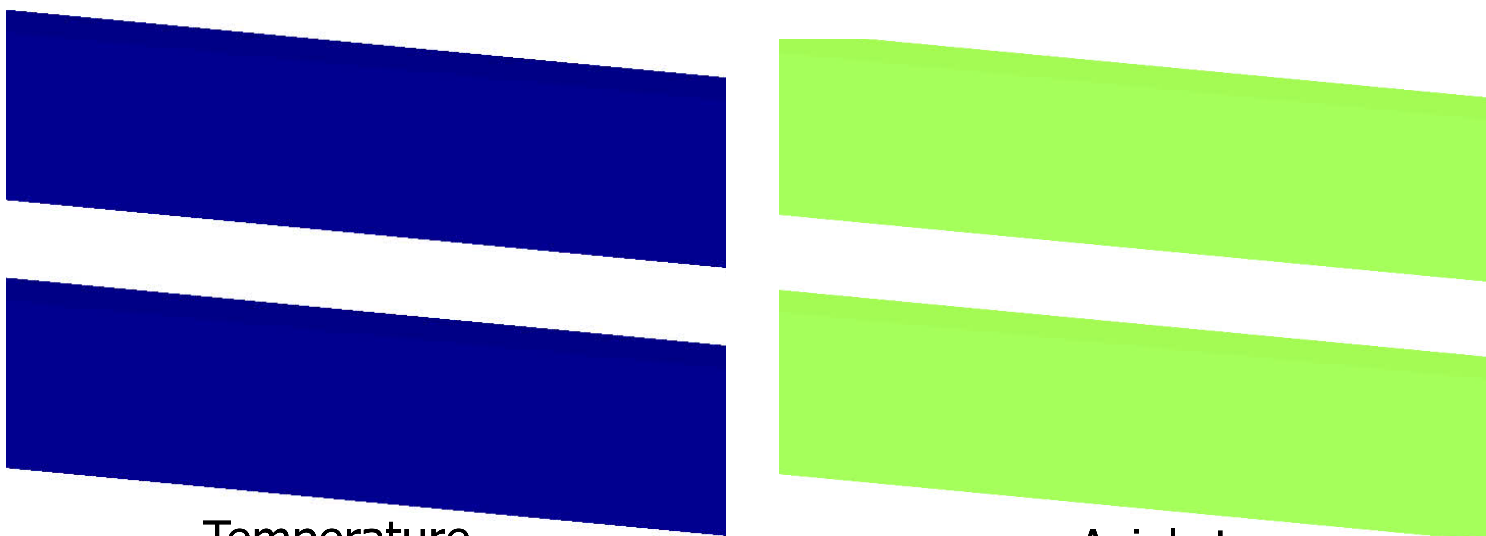


Von Mises stress

Numerical example: Laser-heated beam

- **Case 2:** Moving sharp flux





GFEM^{gl}

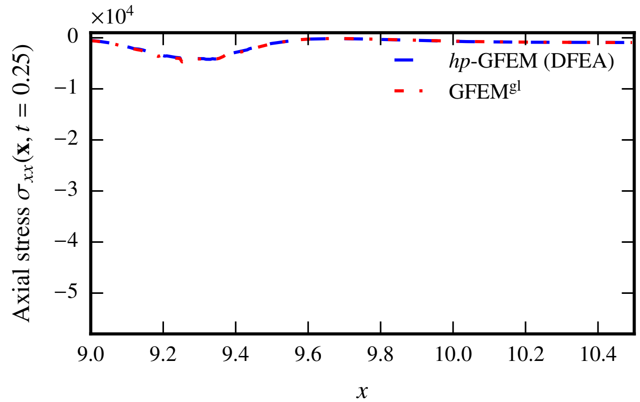
hp-GFEM

Temperature

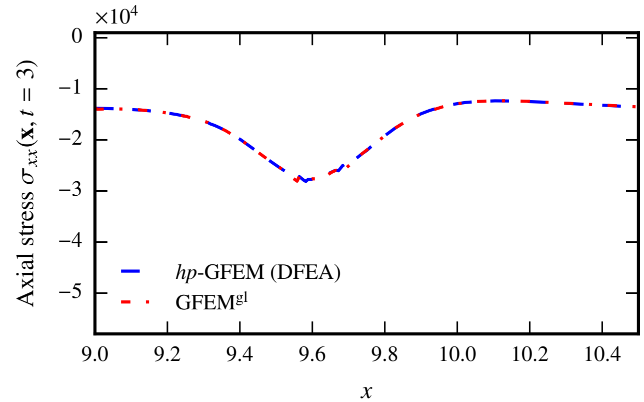
Axial stress

Numerical example: Laser-heated beam

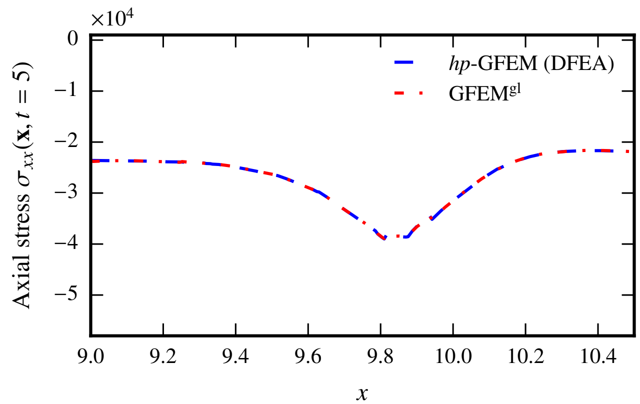
- GFEM^{gl} vs. *hp*-GFEM solutions in time: Axial stress



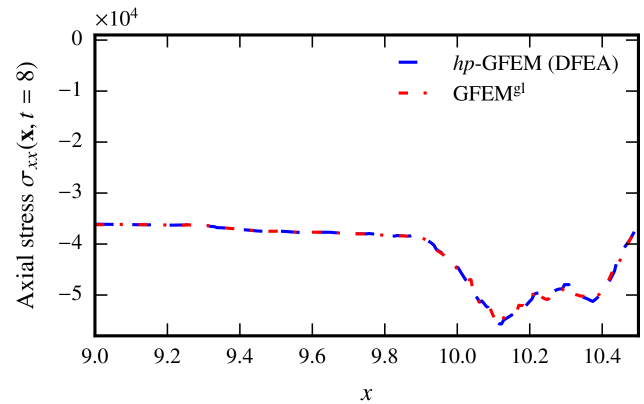
t = 0.25



t = 3.0



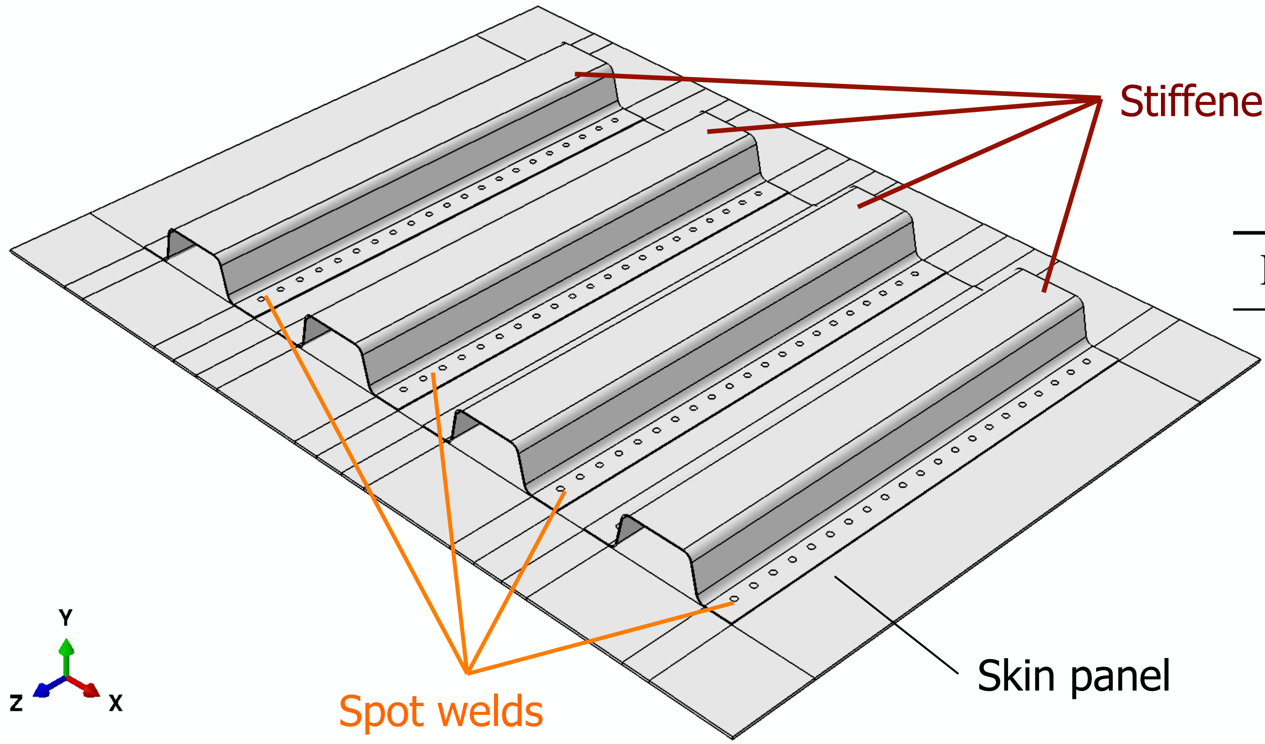
t = 5.0



t = 8.0

GFEM^{gl} Solution of a Hat-Stiffened Panel

- Stiffened panel with 168 spot welds*



Ti-6242S properties:

Property	Value	Units
E	17100	ksi
ν	0.325	–
σ_y	152.0	ksi
α	4.28×10^{-6}	$°F^{-1}$
κ	0.8755	$\frac{\text{ft-lbf}}{\text{s-in-°F}}$
ρc	14.04	$\frac{\text{ft-lbf}}{\text{in}^3\text{-°F}}$

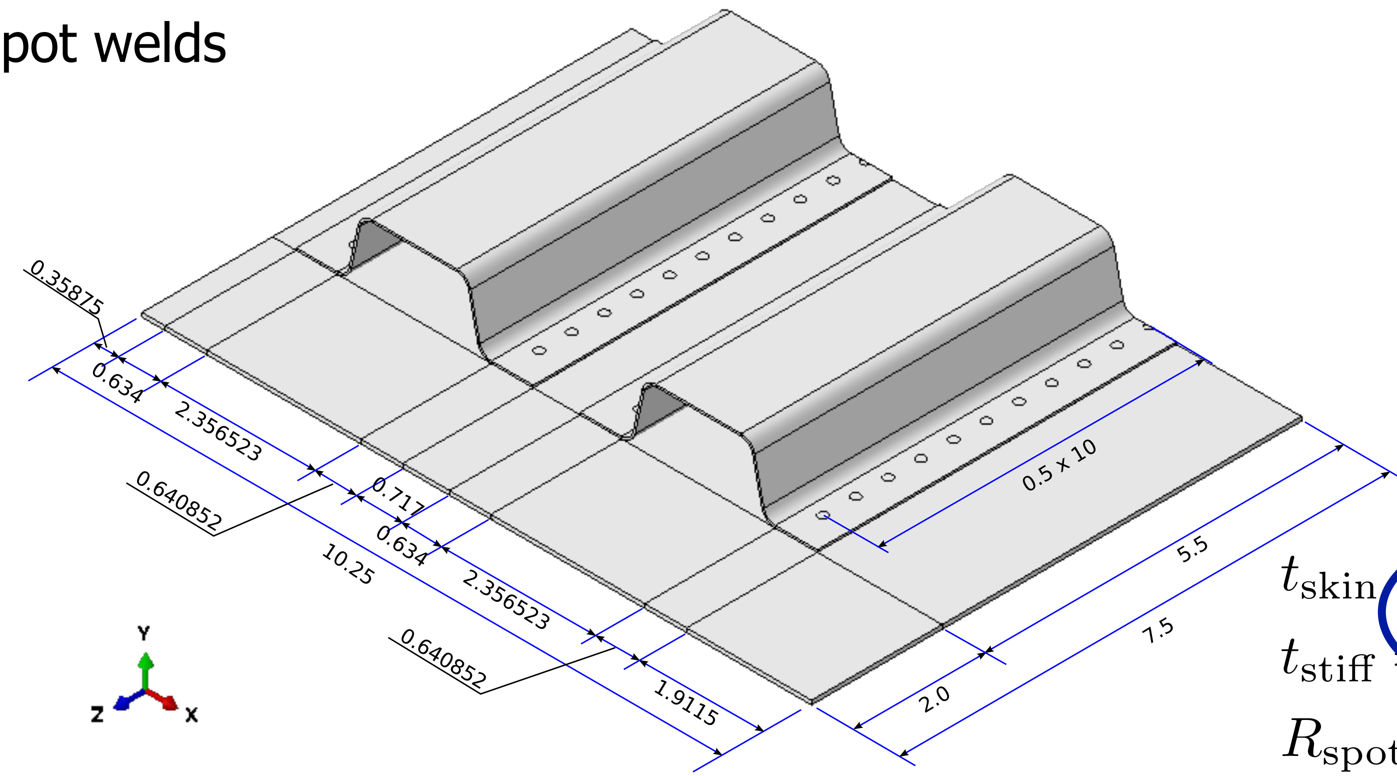
- Represent spot welds using global-local enrichments
- Use a *coarse mesh* at global scale

*Panel geometry and properties courtesy of Air Force Research Laboratory, OH, USA

GFEM^{gl} Solution of a Hat-Stiffened Panel

Case 1: Mechanical load only: Uniform pressure on skin panel

- Linear elastic response
- Use symmetry properties to reduce problem size
- 44 spot welds



Problem dimensions (inches)

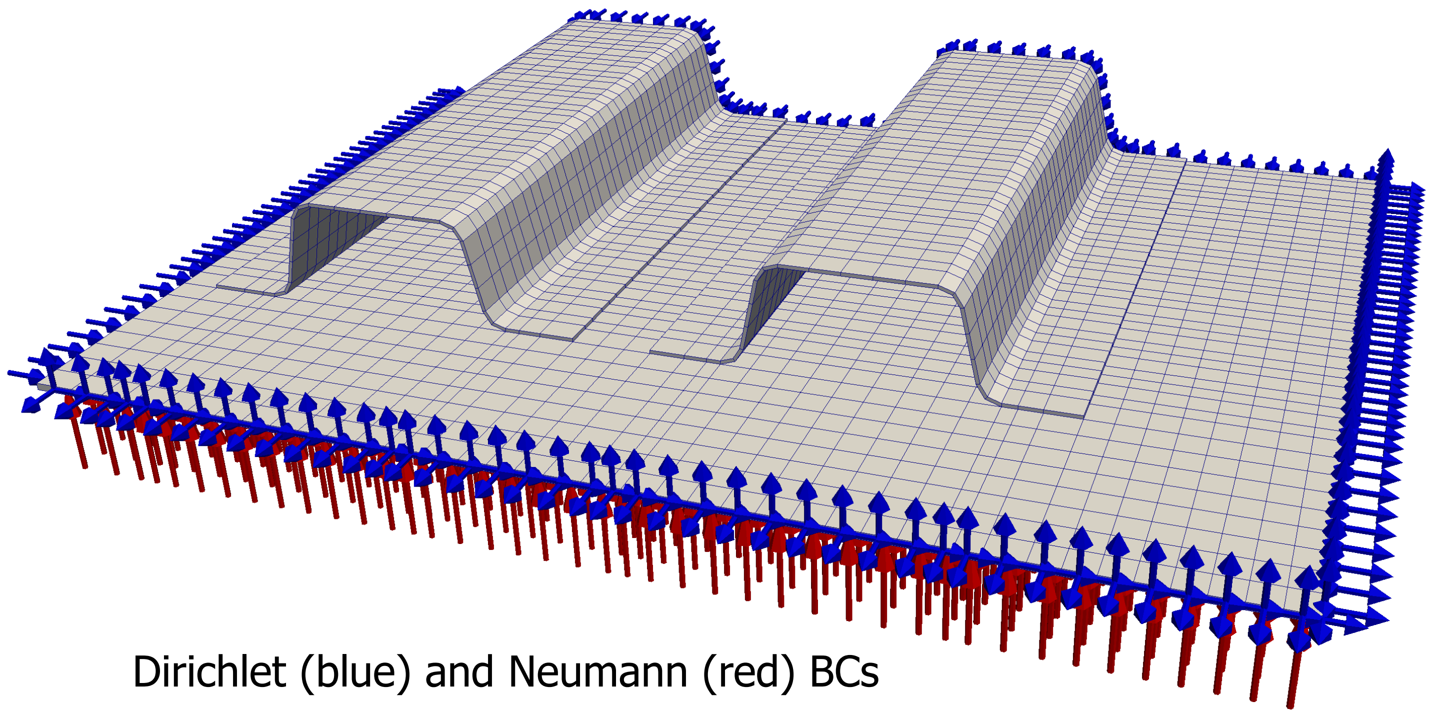
$$t_{\text{skin}} = 0.065\,in$$

$$t_{\text{stiff}} = 0.032\,in$$

$$R_{\text{spot}} = 0.07\,in$$

GFEM^{gl} Solution of a Hat-Stiffened Panel

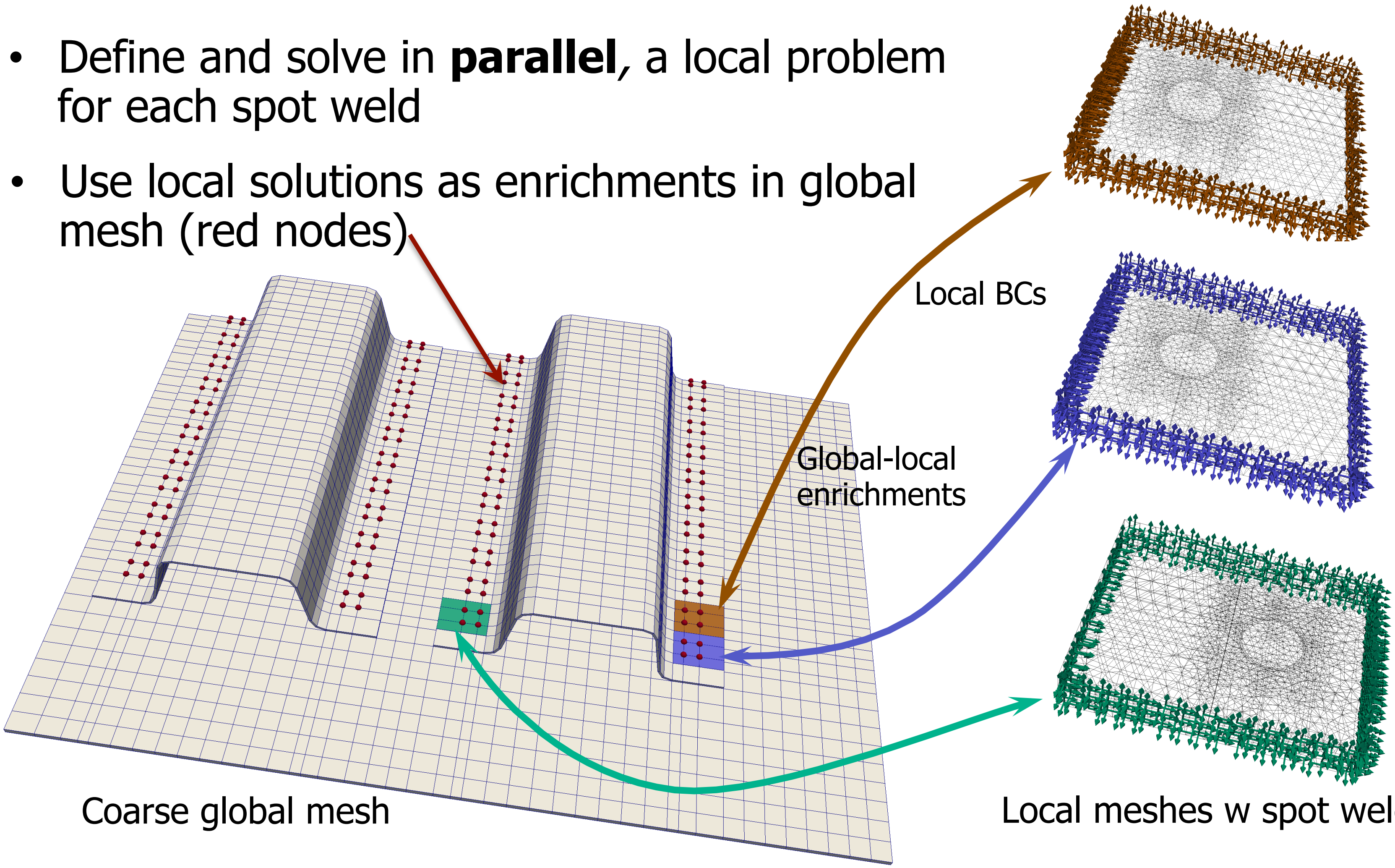
- Global mesh with hexahedron elements: Spot welds are *not* discretized at this scale



Dirichlet (blue) and Neumann (red) BCs

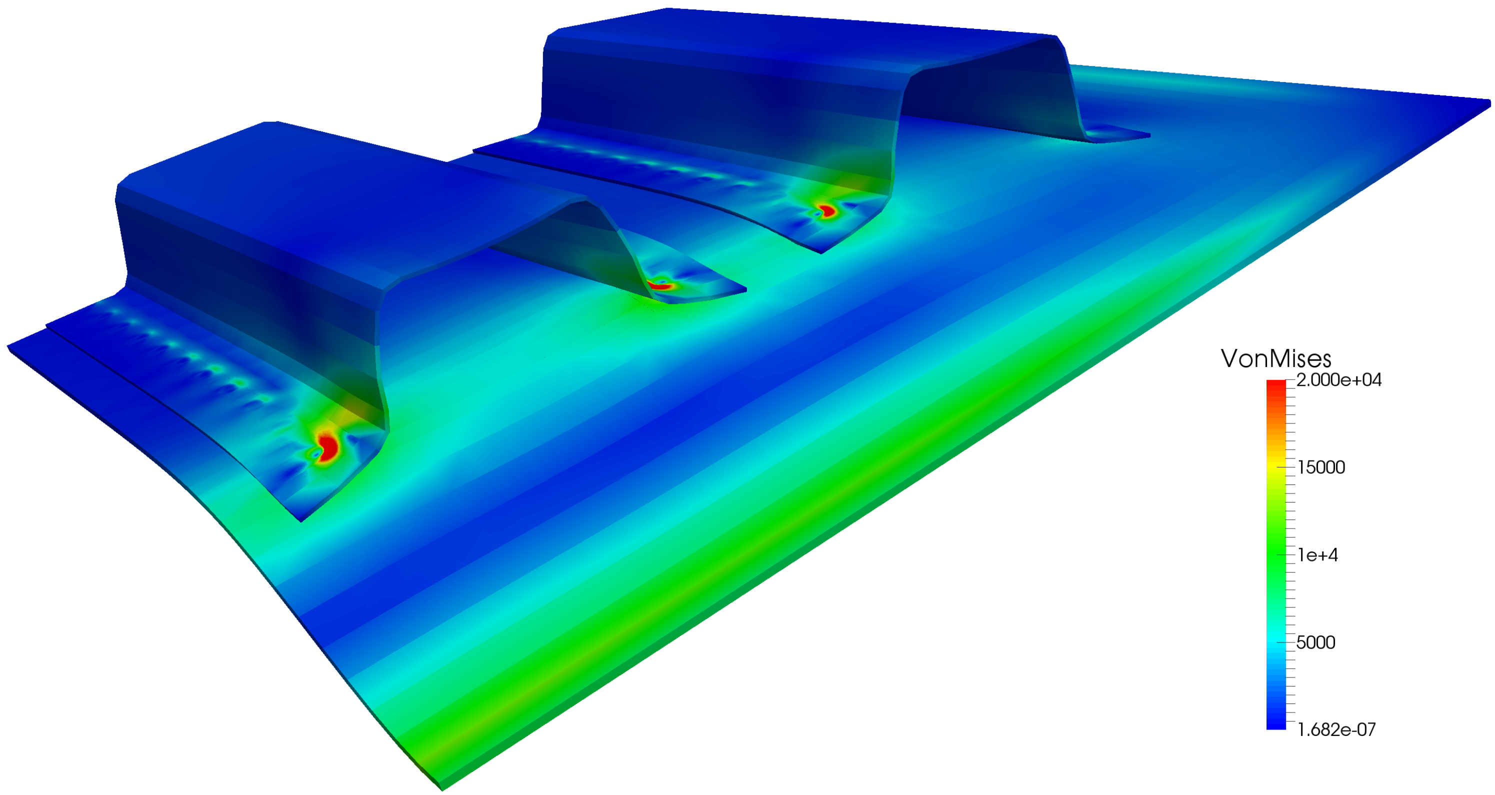
GFEM^{gl} Solution of a Hat-Stiffened Panel

- Global problem provides BCs for local problems
- Define and solve in **parallel**, a local problem for each spot weld
- Use local solutions as enrichments in global mesh (red nodes)



GFEM^{gl} Solution of a Hat-Stiffened Panel

- GFEM^{gl} results: Deformed configuration and von Mises stress

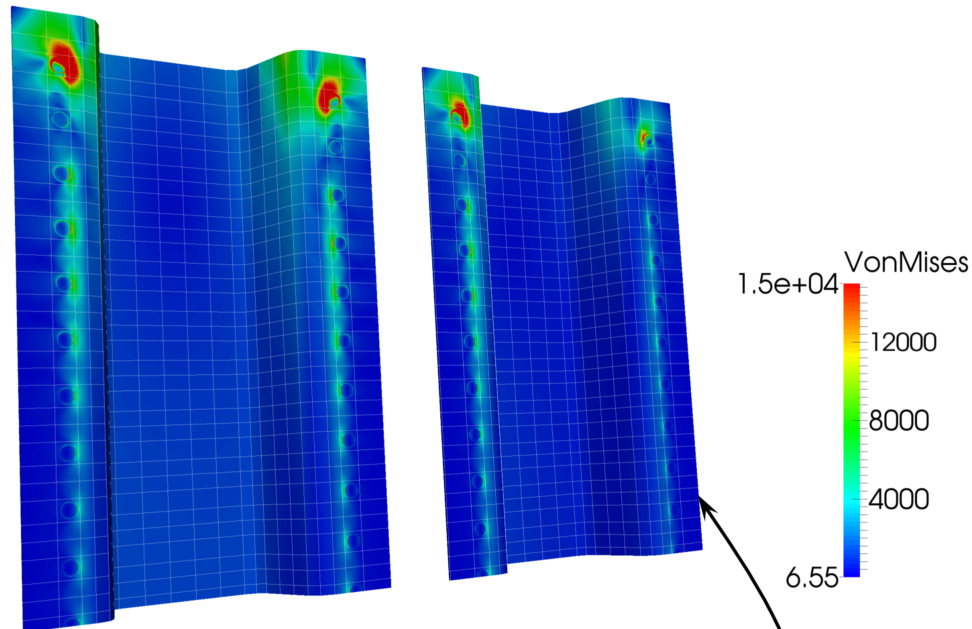


Enriched global problem, deformed shape and Von Mises stress

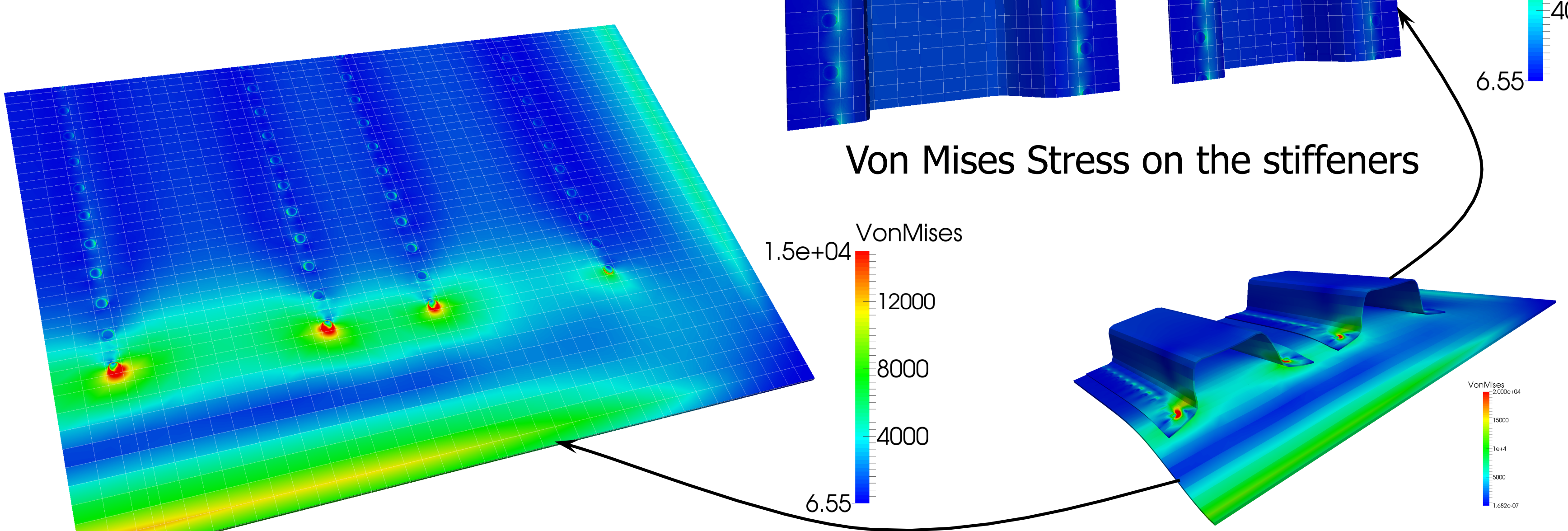
GFEM^{gl} Solution of a Hat-Stiffened Panel

- GFEM^{gl} results: Von Mises stress

Localized stress fields are well captured on a coarse global (HEX8) mesh



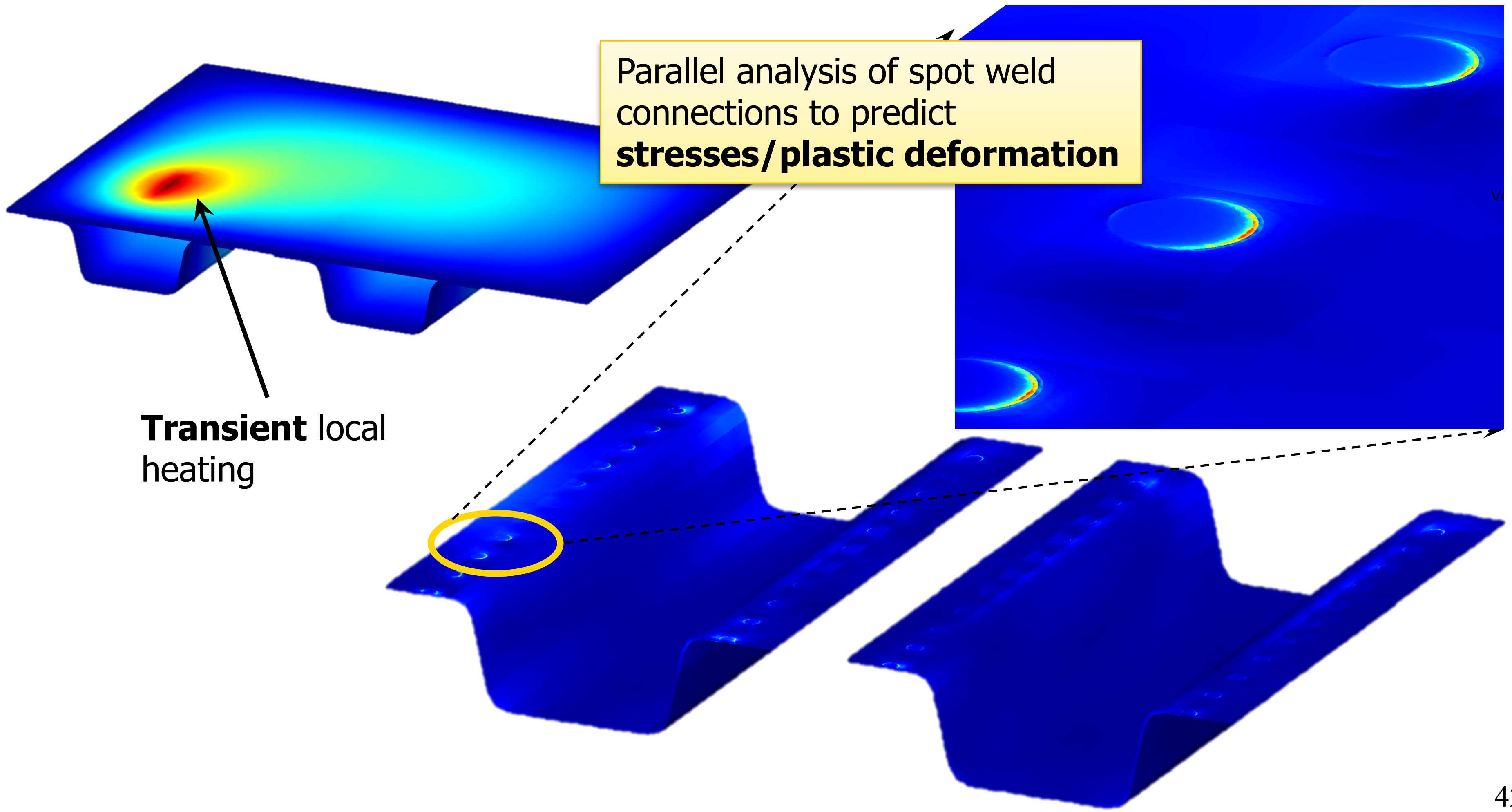
Von Mises Stress on the stiffeners



Von Mises Stress on the skin panel

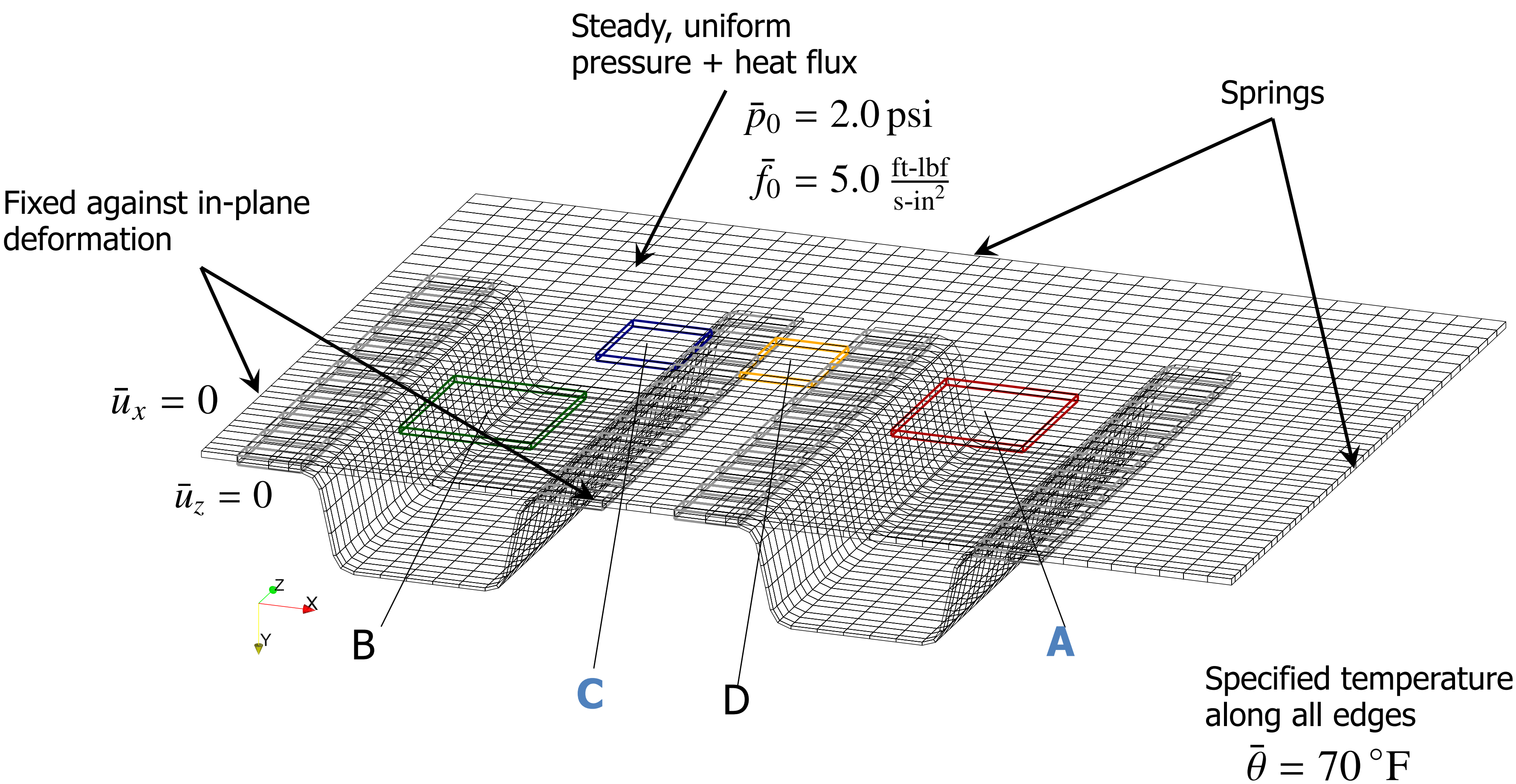
GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Case 2:** Transient nonlinear thermo-mechanical analysis



GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Case 2:** Boundary conditions

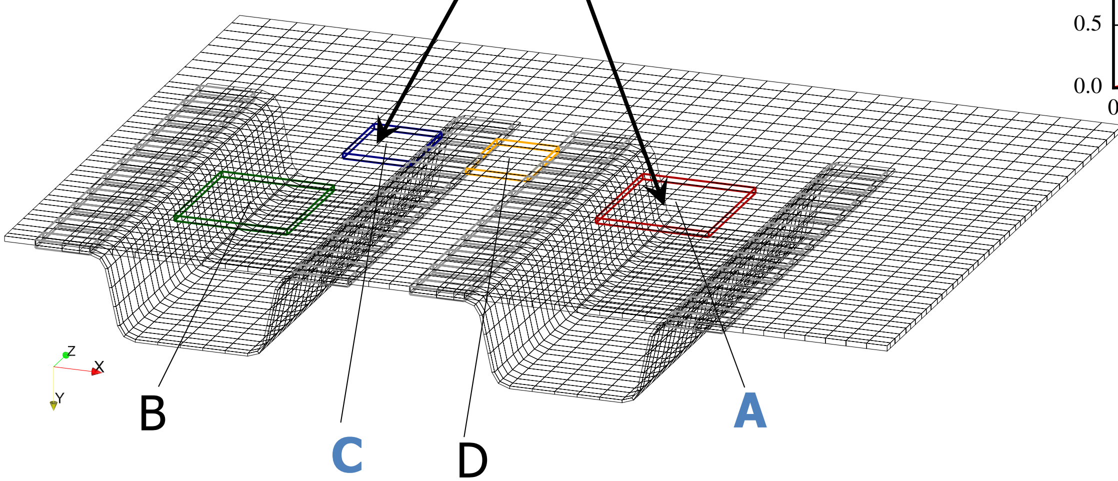


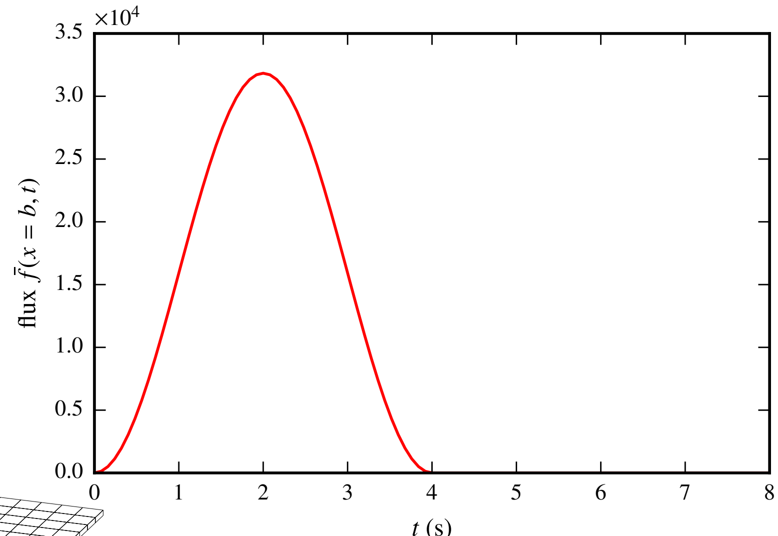
GFEM^{gl} Solution of a Hat-Stiffened Panel

Goal: Find critical location of a localized thermal load

- *Localized, transient heat flux* in Regions A and C:

$$\bar{f}(\mathbf{x}, t) = \frac{I_0}{2\pi a^2} g(t) h(z) \exp\left(\frac{-(x-b)^2}{2a^2}\right) + \bar{f}_0$$

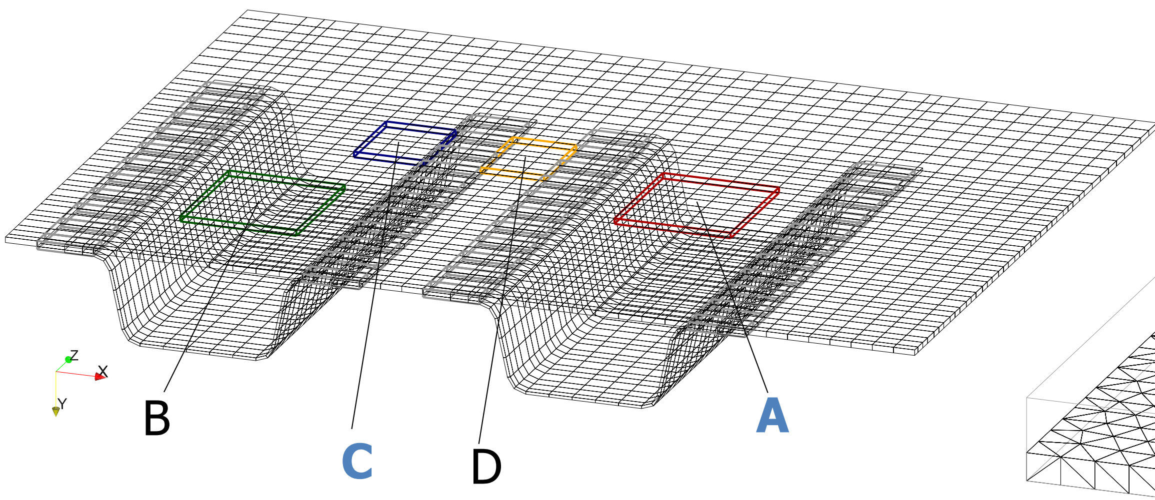




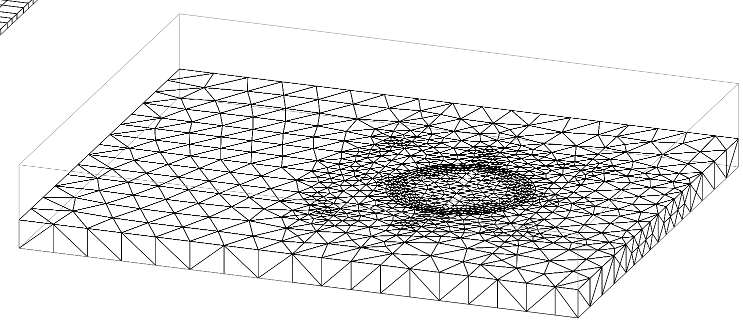
Temporal variation of peak (Gaussian) flux:

GFEM^{gl} Solution of a Hat-Stiffened Panel

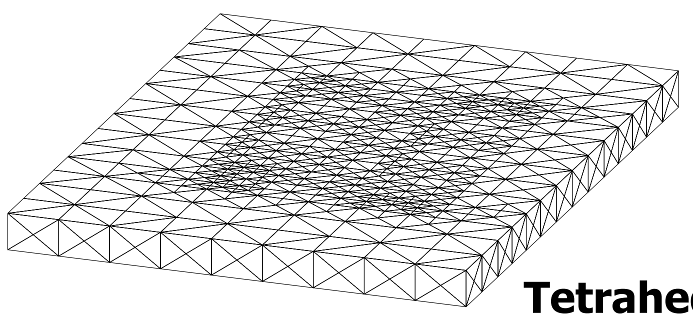
- GFEM^{gl} global + local meshes
- Same global mesh as before



Coarse, global hexahedral mesh (capture structural geometry)



Typical tetrahedral weld mesh (capture local stress concentrations/plasticity in welds)



Tetrahedral local region mesh (capture sharp heat flux)

GFEM^{gl} Solution of a Hat-Stiffened Panel

- Panel/stiffeners connected by series of 44 spot welds

GFEM^{gl} global + local problem sizes

		Problem size (dofs)	
		Heat transfer	Thermoplasticity
Initial global		27,888	209,160
Enriched global (A)		28,480	210,936
Enriched global (C)		28,436	210,804
Local	Full spot weld	7,966	95,592
	Half-spot weld	4,179	50,148
	Region A	5,807	69,684
	Region C	5,965	71,580

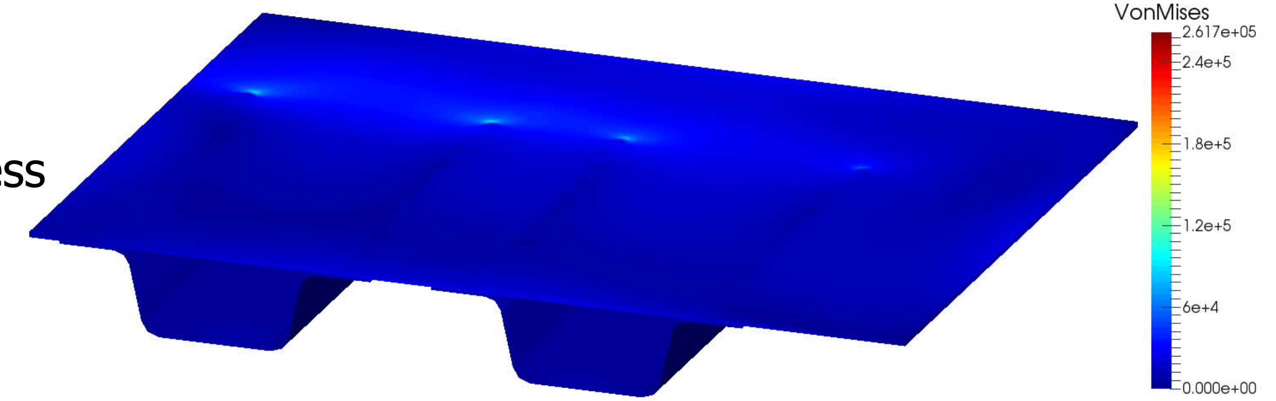
Only ~1,000 extra global dofs from global–local enrichments (*< 1% increase*)

Estimated *hp*-GFEM/hp-FEM (direct analysis) problem size ≈ 4.5 million dofs

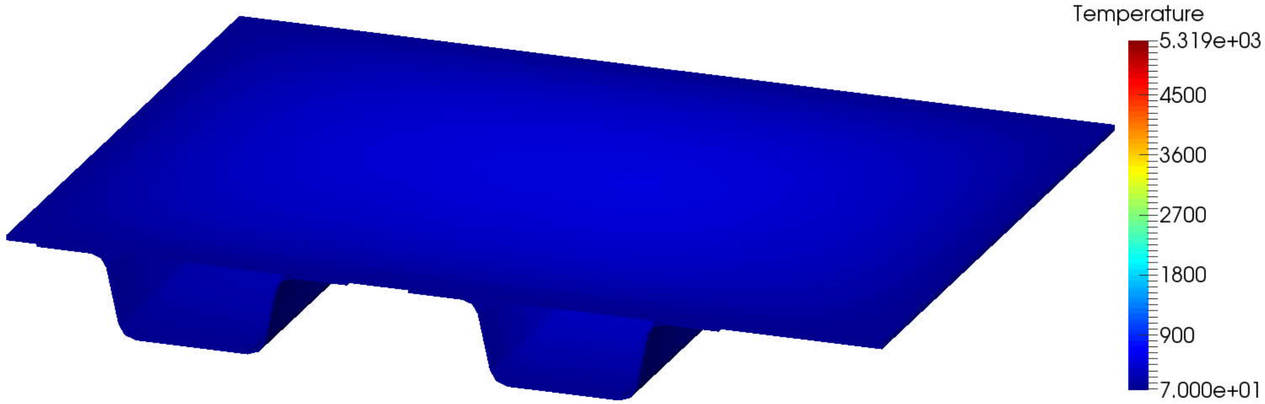
GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Region A**

Von Mises stress



Temperature

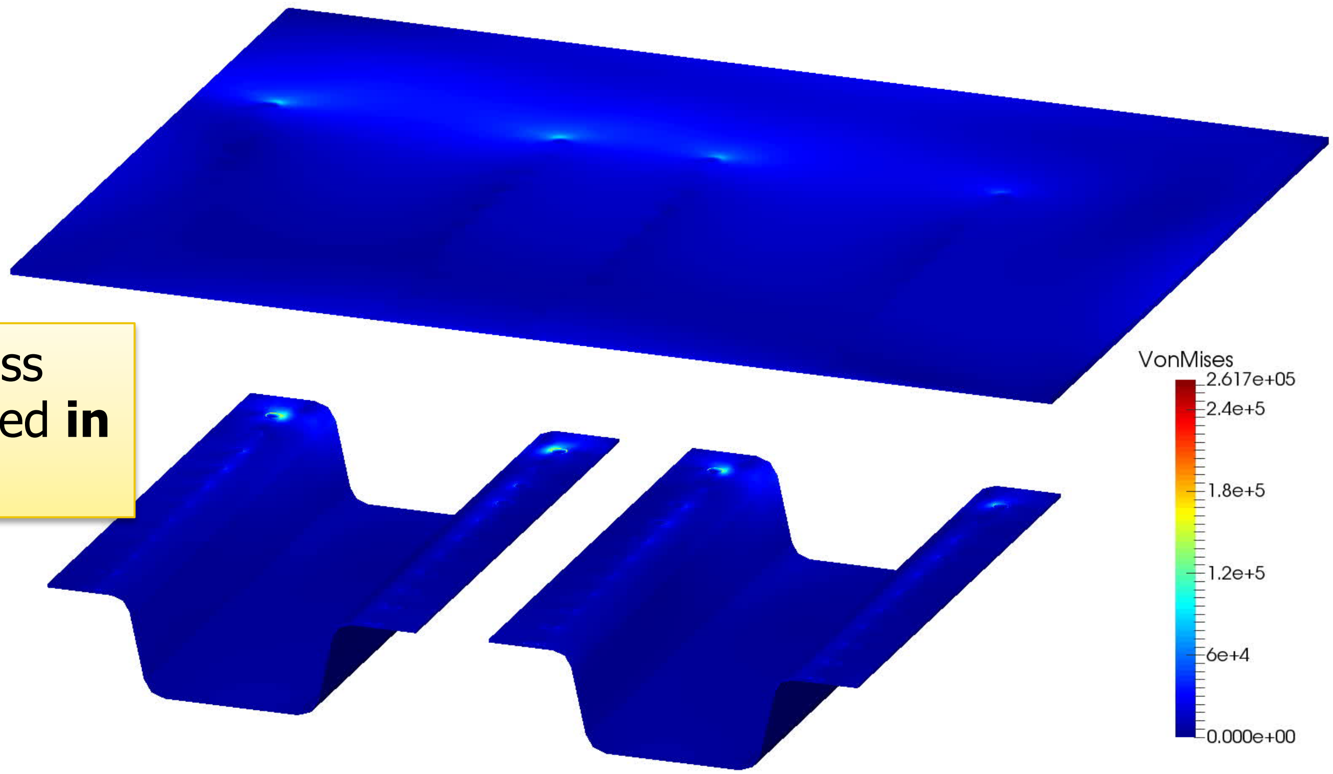


Movie

GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Region A**
  - Von Mises stress (panel + stiffeners)

Local thermal stress effects concentrated **in panel**

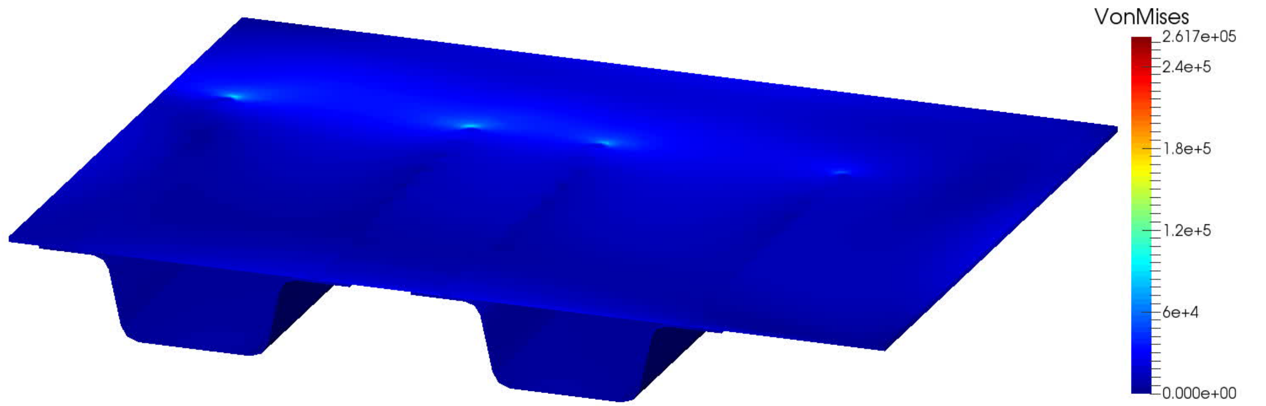


Movie

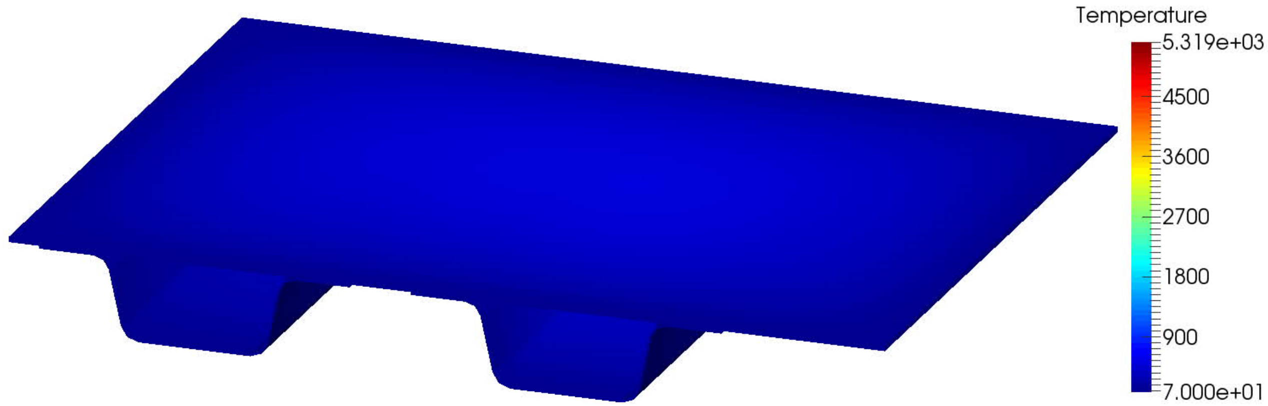
GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Region C**

Von Mises stress



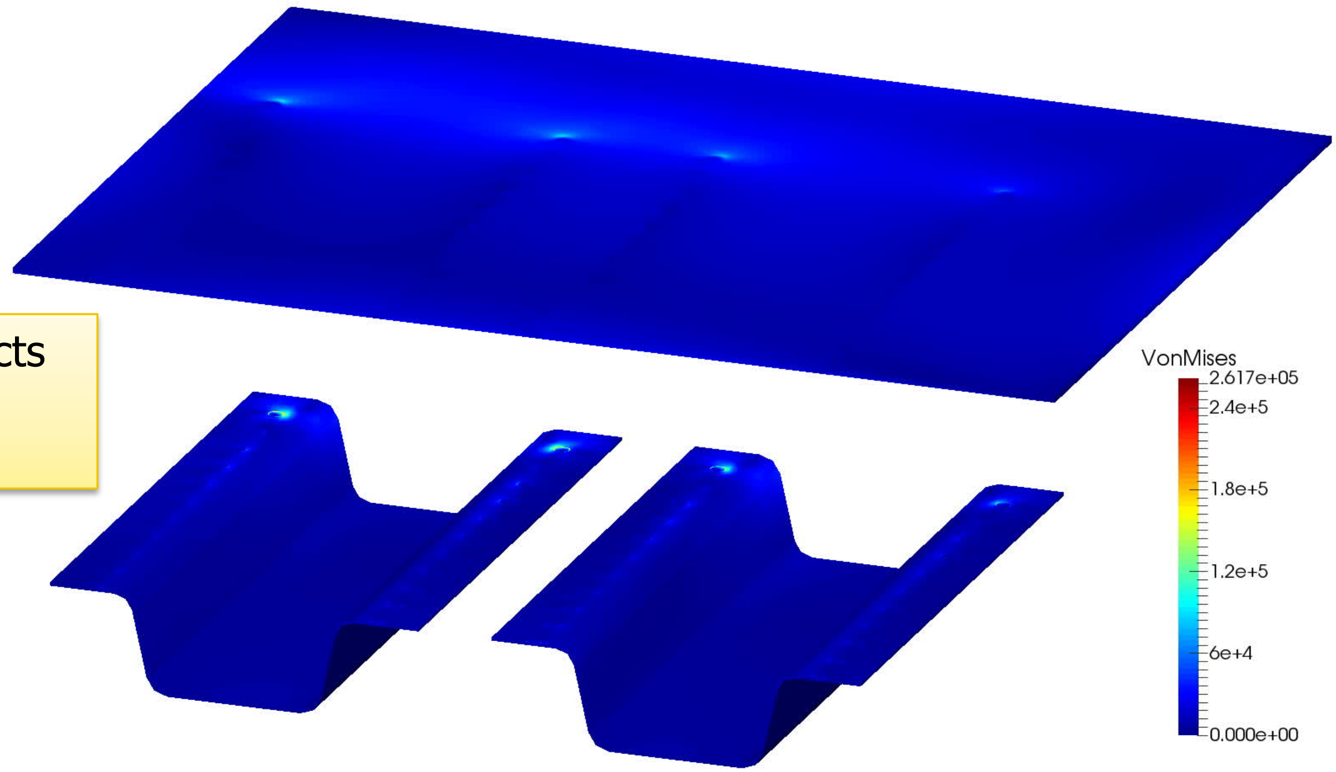
Temperature



Movie

GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Region C**
  - Von Mises stress (panel + stiffeners)

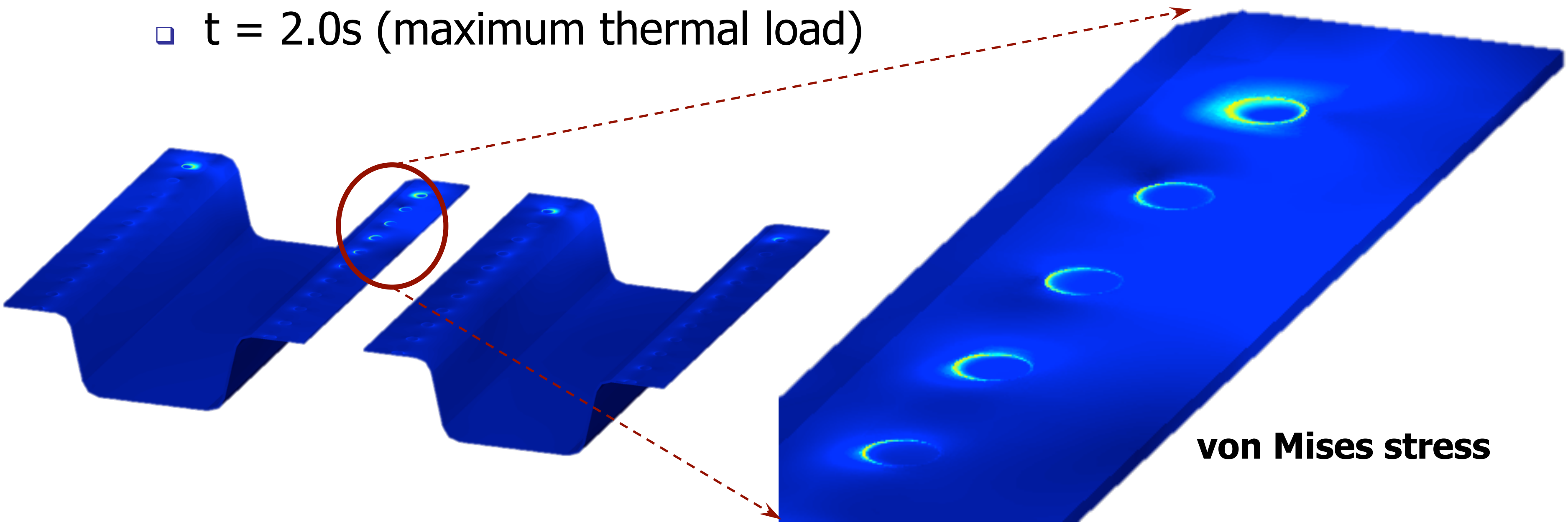


Local thermal stress effects in panel shielded by proximity to **stiffeners**

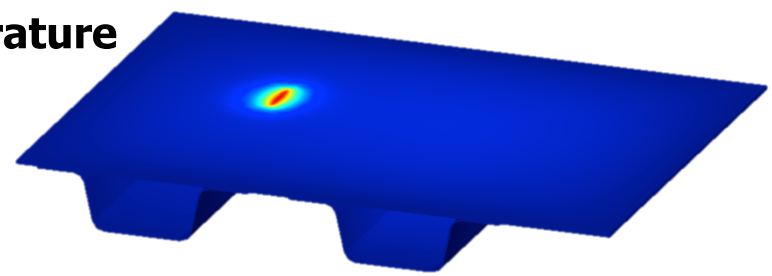
Movie

GFEM^{gl} Solution of a Hat-Stiffened Panel

- **Region C**
  - t = 2.0s (maximum thermal load)



Temperature

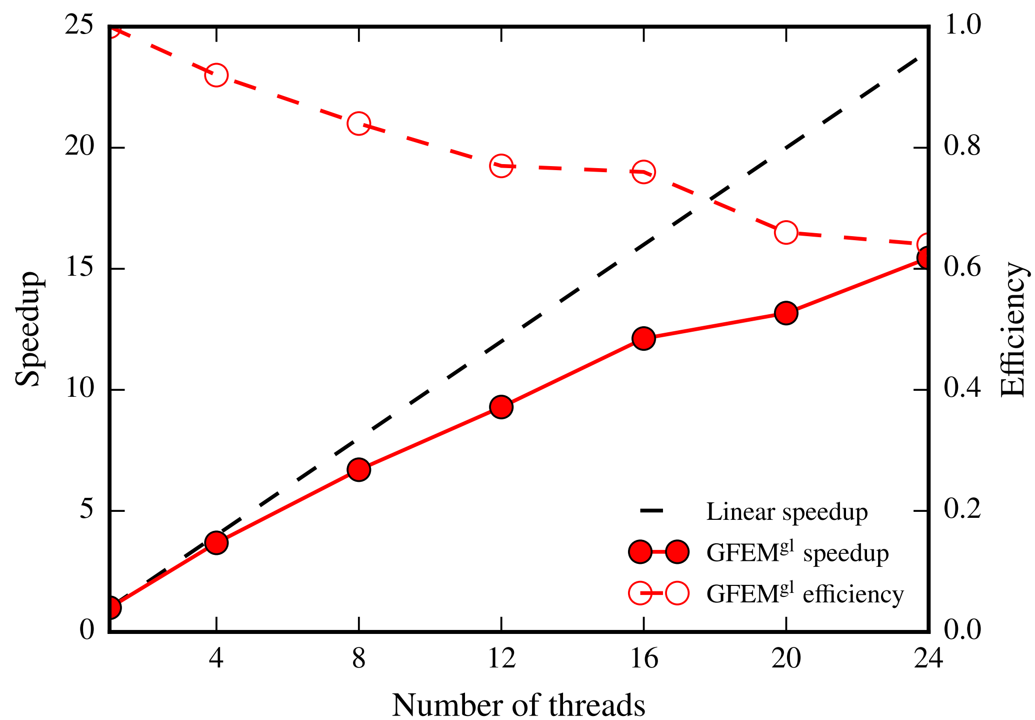


Localized spot weld stresses in vicinity of sharp heating

GFEM^{gl} Solution of a Hat-Stiffened Panel

- GFEM^{gl} parallel performance
  - Single time/load step—all solution phases (local + enriched global) considered
  - Up to 24 CPUs

- Good speedup on small number of threads
- Efficiency deteriorates as number of CPUs increases (expected)

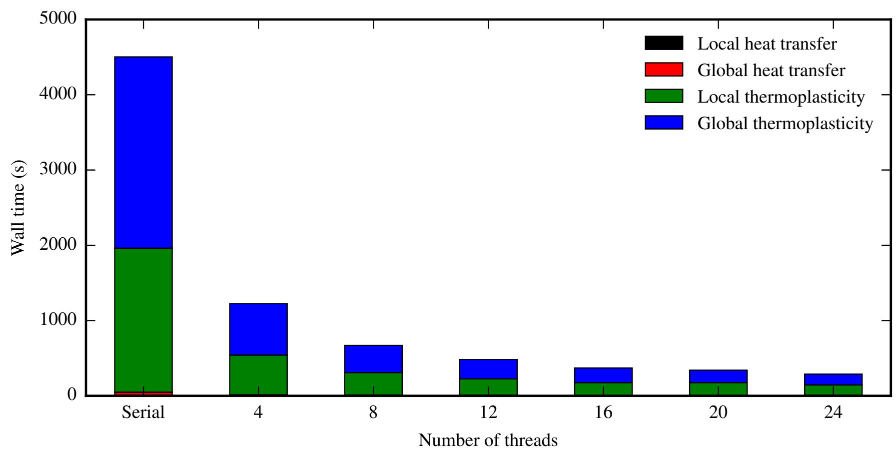


Number of local problems ≈ number of threads—difficult to achieve good **load balance**

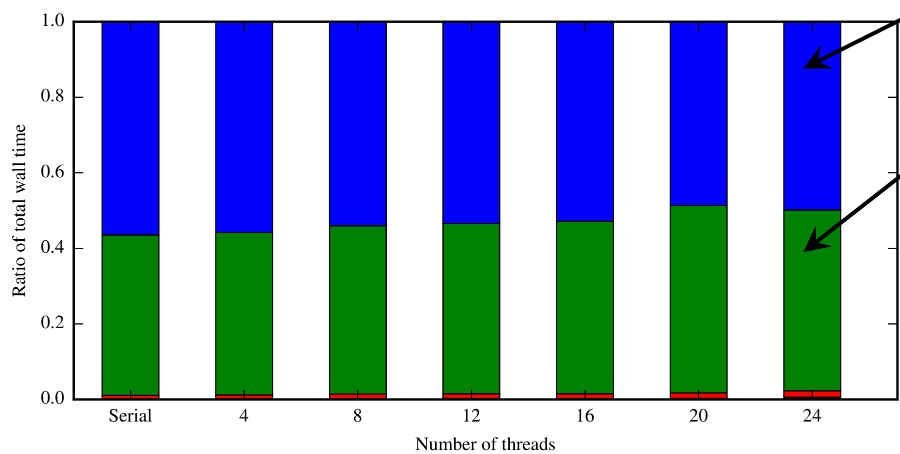
GFEM^{gl} Solution of a Hat-Stiffened Panel

- GFEM^{gl} parallel performance
  - Time spent in each solution phase vs. number of parallel threads:

Total solution wall time



Fraction of total solution time



Enriched global assembly/solution

Local assembly/ solution

Wall times spent in enriched/local problems scale ~uniformly—no bottlenecks!

Summary

- GFEM^{gl} for large, nonlinear, coupled thermo-structural problems exhibiting phenomena spanning multiple spatial scales of interest

- Time-dependent global–local enrichments for capturing nonlinear (elasto-plastic) effects at disparate structural scales

- Fine-scale problems parallelizable; efficiently resolve localized plasticity *at the fine scale*, maintain coarse, global structural mesh

Acknowledgements

Jongheon Kim, Haoyang Li and Patrick O’Hara

caduarte@illinois.edu

http://gfem.cee.illinois.edu/