A New Generalized FEM for Two-Scale Simulations of Propagating Cohesive Fractures in 3-D

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Motivation: Multiscale Structural Analysis

The US Air Force has expended six decades and untold resources in attempts to field a reusable hypersonic vehicle*

Scientific challenge:

“An inability to computationally capture the material evolution and degradation within a structural component”


*[T. G. Eason et al., 2013]
Motivation: Multiscale Structural Analysis

- Thermo, mechanical and acoustic loads on hypersonic aircrafts lead to highly localized non-linear 3-D stress fields: Finite element models with fine meshes are required.

3-D FEM: Large aspect ratio of elements may lead to numerical instabilities during analysis [Sobotka et al., 2013]
Motivation: Multiscale Structural Analysis

- Highly localized non-linear 3-D effects
- Most of the structure remains linear elastic

Q: How to efficiently capture these localized non-linear 3-D effects?
Q: How to avoid numerical stability issues caused by aspect ratio of elements?

Strategy: A Generalized FEM for multiscale structural analysis

[Sobotka et al., 2013]
Outline

- Motivation
- Generalized finite element methods: Basic ideas
- Bridging scales with GFEM:
  - Global-local enrichments for localized non-linearities
- Numerical examples
- Conclusions
Generalized Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

\[ S_{GFEM} = S_{FEM} + S_{ENR} \]

Low order FEM space \quad Enrichment space with functions related to the given problem

\[ S_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R} \]

\[ S_{ENR} = \sum_{\alpha \in I^e_h \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha} \]

\[ L_{\alpha i} \in \chi_\alpha(\omega_\alpha) \]

Enrichment function \quad Patch space
Generalized Finite Element Method

\[ S_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha} \]

\[ \phi_{\alpha i}(\mathbf{x}) = \varphi_\alpha(\mathbf{x}) L_{\alpha i}(\mathbf{x}) \quad \sum_{\alpha} \varphi_\alpha(\mathbf{x}) = 1 \]

- Allows construction of shape functions incorporating a-priori knowledge about solution

[Oden, Duarte & Zienkiewicz, 1996]

Discontinuous enrichment [Moes et al., 1999]
Multiscale Structural Analysis

- Highly localized non-linear 3-D effects
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Q: How to efficiently capture these localized non-linear 3-D effects?
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Strategy: A Generalized FEM for multiscale structural analysis

[Sobotka et al., 2013]
Bridging Scales with Global-Local Enrichment Functions*

- Enrichment functions computed from solution of local boundary value problems: Global-Local enrichment functions

- **Idea:** Use available numerical solution at a simulation step to build shape functions for next step (quasi-static, transient, non-linear, etc.)

- Enrichment functions are produced numerically on-the-fly through a global-local analysis

- Use a **coarse** mesh enriched with Global-Local (GL) functions

- $\text{GFEM}^{\text{gl}} = \text{GFEM}$ with global-local enrichments

* [Duarte et al. 2005]
Global-Local Enrichments for Problems with Localized Non-Linearities

- **Model Problem**: Simulation of propagating cracks using cohesive fracture models

\[ \begin{align*}
\text{Find } \mathbf{u} &\in H^1(\Omega_G), \text{ such that } \forall \ \delta \mathbf{u} \in H^1(\Omega_G) \\
&\int_{\Omega_G} \nabla^s (\delta \mathbf{u}) : \mathbf{\sigma} (\mathbf{u}) \ dV + \int_{\Gamma^\text{coh}} \delta [\mathbf{u}] \cdot t^\text{coh} ([\mathbf{u}]) \ dS + \eta \int_{\Gamma^u_G} \delta \mathbf{u} \cdot \mathbf{u} \ dS \\
&\quad = \int_{\Omega_G} \delta \mathbf{u} \cdot \mathbf{b} \ dV + \int_{\Gamma^t_G} \delta \mathbf{u} \cdot \mathbf{t} \ dS + \eta \int_{\Gamma^u_G} \delta \mathbf{u} \cdot \mathbf{\bar{u}} \ dS
\end{align*} \]
Global-Local Enrichments for Problems with Localized Non-Linearities

- Three-Point Bending Beam

- Typical FEM discretization [Park et al. 2008]

Goals:
- Solve problem on a coarse global mesh.
- Non-linear iterations at fine scales only.
Global-Local Enrichments for Problems with Localized Non-Linearities

Let \( u^n_G \in S^n_G(\Omega) \) be a GFEM approximation of global problem at load step \( n \). Global-local enrichments are used in the definition of \( S^n_G(\Omega) \).

Compute a rough and cheap estimate of global solution at next load step.

Define \( u^{n+1}_{G,0} = \frac{n+1}{n} u^n_G \)

\( u^{n+1}_{G,0} \) is used to prescribe boundary conditions for a non-linear local problem as defined next.
Global-Local Enrichments for Problems with Localized Non-Linearities

- Solve following non-linear local problem at load step n+1 using, e.g., $hp$-GFEM

\[
\begin{align*}
\int_{\Omega_L} \nabla^s (\delta u_L^{n+1}) : \sigma (u_L^{n+1}) \, dV + \int_{\Gamma_{coh}} \delta [u_L^{n+1}] \cdot t_L^{coh} ([u_L^{n+1}]) \, dS + \eta \int_{\Gamma_L \cap \Gamma_G^{\nu}} \delta u_L^{n+1} \cdot u_L^{n+1} \, dS \\
+ \kappa \int_{\Gamma_L \setminus (\Gamma_L \cap \Gamma_G^{\nu} \cup \Gamma_G^t)} \delta u_L^{n+1} \cdot u_L^{n+1} \, dS = \int_{\Omega_L} \delta u_L^{n+1} \cdot b \, dV + \int_{\Gamma_L \cap \Gamma_G^t} \delta u_L^{n+1} \cdot \bar{t} \, dS \\
+ \eta \int_{\Gamma_L \cap \Gamma_G^{\nu}} \delta u_L^{n+1} \cdot \bar{u} \, dS + \int_{\Gamma_L \setminus (\Gamma_L \cap \Gamma_G^{\nu} \cup \Gamma_G^t)} \delta u_L^{n+1} \cdot [t_G^{n+1} (u_G^{n+1}) + \kappa u_G^{n+1}] \, dS
\end{align*}
\]
Global-Local Enrichments for Problems with Localized Non-Linearities

- Global space enriched with non-linear local solution

\[ \phi^{n+1}_\alpha (\mathbf{x}) = \varphi_\alpha (\mathbf{x}) u_L^{n+1} (\mathbf{x}) \]

\[ u_G^{n+1} (\mathbf{x}) \in S_G^{n+1} (\Omega_G) = S_G^{\text{FEM}} + \{ \varphi_\alpha u^{\text{gl},n+1}_\alpha, \alpha \in \mathcal{I}_\text{gl} \} \]

where \( u^{\text{gl},n+1}_\alpha (\mathbf{x}) = \begin{cases} u_\alpha u^{n+1,<0>}_L (\mathbf{x}) \\ v_\alpha u^{n+1,<1>}_L (\mathbf{x}) \\ w_\alpha u^{n+1,<2>}_L (\mathbf{x}) \end{cases} \), \( u_\alpha, v_\alpha, w_\alpha \in \mathbb{R} \)

- Discretization spaces updated on-the-fly with global-local enrichment functions
Global-Local Enrichments for Problems with Localized Non-Linearities

• On-the-fly updating of global-local enrichment functions during the non-linear iterative solution process
Global-Local Enrichments for Problems with Localized Non-Linearities

• Global space at load step $n+1$

$$u_{G}^{n+1}(\mathbf{x}) \in \mathbb{S}_{G}^{n+1}(\Omega_{G}) = \mathbb{S}_{G}^{\text{FEM}} + \mathbb{S}_{G}^{\text{ENR},n+1}$$

where $\mathbb{S}_{G}^{\text{ENR},n+1} = \{ \varphi_{\alpha}u_{\alpha}^{\text{gl},n+1}, \alpha \in \mathcal{I}_{\text{gl}} \}$

• Discretization spaces updated on-the-fly with global-local enrichment functions
• Enrichment space is load dependent
• Dimension of global space does not change but its basis functions do:

Use of vector with global DOFs computed at previous load step is not a robust choice for the initialization of the Newton-Rhapson non-linear iterations at this load step.

$d_{G}^{n}$ and $d_{G}^{m+1}$ represent coefficients of different sets of GFEM shape functions

Classical strategy: Map $u_{G}^{n} \in \mathbb{S}_{G}^{n}(\Omega_{G})$ into $\mathbb{S}_{G}^{n+1}(\Omega_{G})$
Global-Local Enrichments for Problems with Localized Non-Linearities

- **Linear** global problem at load step \( n+1 \)

Find \( \mathbf{u}^{n+1} \in \mathbb{S}^{n+1}_G(\Omega_G) \) such that, \( \forall \; \delta \mathbf{u}^{n+1} \in \mathbb{S}^{n+1}_G(\Omega_G) \)

\[
\int_{\Omega_G} \nabla \mathbf{s}(\delta \mathbf{u}^{n+1}) : \mathbf{\sigma}(\mathbf{u}_G^{n+1}) \; dV + \int_{\Gamma_{coh}} \delta [\mathbf{u}^{n+1}_G] \cdot C^{m+1}_D (\mathbf{u}^{n+1}_L)[\mathbf{u}^{n+1}_G] \; dS + \eta \int_{\Gamma_{ru}} \delta \mathbf{u}^{n+1}_G \cdot \mathbf{u}^{n+1}_G \; dS = \int_{\Omega_G} \delta \mathbf{u}^{n+1}_G \cdot \mathbf{b} \; dV + \int_{\Gamma_{G}} \delta \mathbf{u}^{n+1}_G \cdot \bar{\mathbf{t}} \; dS + \eta \int_{\Gamma_{ru}} \delta \mathbf{u}^{n+1}_G \cdot \bar{\mathbf{u}} \; dS
\]

The cohesive secant matrix is given by

\[
C^{n+1} = \begin{bmatrix}
C^{m+1, <m_0>}_D & 0 & 0 \\
0 & C^{m+1, <m_1>}_D & 0 \\
0 & 0 & C^{m+1, <m_2>}_D
\end{bmatrix}
\]

where

\[
C^{m+1, <m_t>}_D = \frac{t^{<m_t>}_{coh}(\mathbb{u}^{n+1}_L)}{\mathbb{u}^{n+1}_L^{<m_t>}}, \; t = 0, 1, 2
\]
GFEM$^{gl}$ Algorithm for Cohesive Fractures

- A linear global problem with secant stiffness provided by local problem is solved at every load step.
- All non-linearities are handled at the local problem.
- BCs for local problem can be improved through global-local iterations.

Fine-scale local problem

- BCs
- Enrichments
- Damage parameters

- : nodes enriched by polynomials & Heaviside functions
- : nodes enriched by only polynomials

Interface between local domain boundary and global domain

Fine-scale local problem

- : nodes enriched by polynomials & local solutions
- : nodes enriched by only polynomials
Global-Local Enrichments for Problems with Localized Non-Linearities

Step 60
Global-Local Enrichments for Problems with Localized Non-Linearities

Experiments by [Roesler et al., 2007], 2-D FEM results by [Park et al. 2008]

2-D FEM, $hp$-GFEM, and GFEM$^{gl}$ meshes
Control of Non-Linear Residue of Global Solution

The residue of the original non-linear global problem is evaluated using

$$R_{G}^{n+1} = f_{G,ext}^{n+1}(d_{G}^{n+1}) - f_{G,int}^{n+1}(d_{G}^{n+1}, d_{L}^{n+1})$$

where the internal and external force vectors are given by

$$f_{G,int}^{n+1}(d_{G}^{n+1}, d_{L}^{n+1}) = \int_{\Omega_{G}} B^{T} \sigma(d_{G}^{n+1}) \, dV + \int_{\Gamma_{coh}} \phi^{T} t_{coh}(d_{G}^{n+1}) \, dS + \eta \int_{\Gamma_{u_{G}}} N^{T} u_{G}^{n+1}(d_{G}^{n+1}) \, dS$$

$$f_{G,ext}^{n+1}(d_{L}^{n+1}) = \int_{\Omega_{G}} N^{T} b^{n+1} \, dV + \int_{\Gamma_{t_{G}}} N^{T} \tilde{t}^{n+1} \, dS + \eta \int_{\Gamma_{u_{G}}} N^{T} \tilde{u}^{n+1} \, dS$$

The magnitude of residue relative to external forces is measured using

$$W_{G}^{n+1} = \frac{R_{G}^{n+1T} d_{G}^{n+1}}{f_{G,ext}^{n+1T} d_{G}^{n+1}}$$
Control of Non-Linear Residue of Global Solution

- Non-linear residue of global solution computed with secant stiffness
- Residue is maximum at load step 37 while limit point is reached at load step 28
- One Newton-Raphason iteration reduces residue to $\sim 10^{-11}$ at all load steps
Control of Non-Linear Residue of Global Solution

If the residue

\[ W_{G}^{n+1} = \frac{R_{G}^{n+1}T}{f_{G,ext}^{n+1}T} d_{G}^{n+1} \]

is above acceptable tolerance:

- Solve non-linear global problem using Newton-Rhapson iterations with a cohesive tangent matrix instead of secant stiffness
- The solution of the global problem computed using secant stiffness provides a robust initialization of the iterative process
Delamination Test: Problem Setup

Bonded double cantilever beam (DCB) specimen

- Isotropic linear elasticity in the bulk
- Linear and Paulino-Park-Roesler (PPR) cohesive models adopted

• $\vec{u}_1 = \vec{u}_2$: mode I, $\vec{u}_1 = 2\vec{u}_2$: mixed mode

• Isotropic linear elasticity in the bulk
• Linear and Paulino-Park-Roesler (PPR) cohesive models adopted
GFEM\textsuperscript{gl} Algorithm for Cohesive Fractures

Coarse-scale global problem

- : nodes enriched by polynomials & local solutions
- : nodes enriched by only polynomials

Fine-scale local problem

- : nodes enriched by polynomials & Heaviside functions
- : nodes enriched by only polynomials

BCs, Enrichments, Damage parameters

blending elements

Interface between local domain boundary and global domain
Delamination Test

- Solution for mixed mode case, linear cohesive law
- Stress component normal to crack is shown
Delamination Test

Computed reaction versus imposed average displacement: Mixed mode case

- Linear cohesive law
- PPR cohesive law
Delamination Test

- Non-linear residue of global solution computed with secant stiffness

- Load steps corresponding to the limit point are *not* the same as those with maximum residue
- One Newton-Raphason iteration reduces residue to $\sim 10^{-11}$ at all load steps
Conclusions and Outlook

- Proposed GFEMgl enables effective resolution of highly localized non-linearities on coarse, structural-scale finite element meshes
- Neither global nor local meshes need to fit crack surface
- Method can be used with virtually any cohesive model and does not require a-priori knowledge about the exact solution of the problem
- Non-linear Newton-Rhapson iterations are in general performed only at local problems used in the computation of enrichment functions
- Solutions with comparable accuracy to those provided by adaptive GFEM are obtained with significantly fewer degrees of freedom at the global problem
- Amenable to non-intrusive integration with existing FEA software: Transition to Labs and Industries
- Extensions to fluid-driven cracks (hydraulic fractures) are underway
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