UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

A New Generalized FEM for Two-Scale Simulations of Propagating Cohesive Fractures in 3-D

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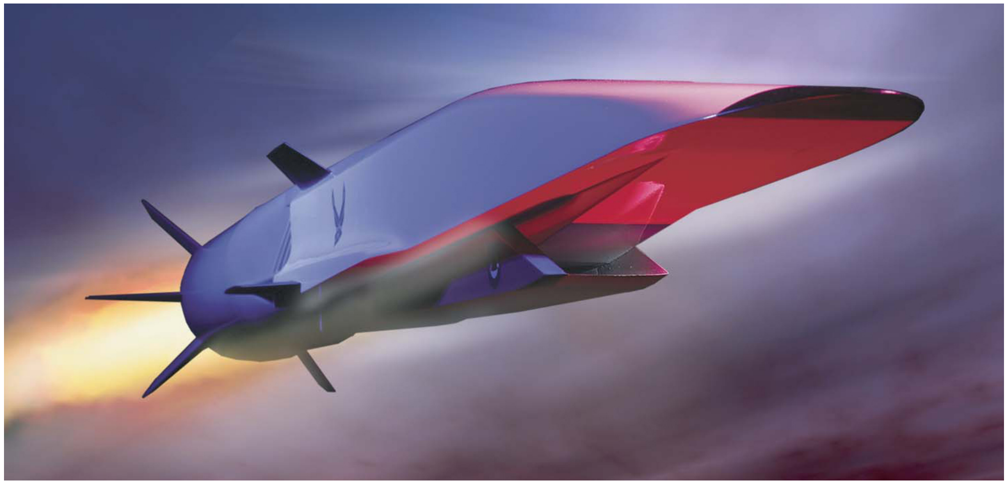
Motivation: Multiscale Structural Analysis

The US Air Force has expended six decades and untold resources in attempts to field a reusable hypersonic vehicle*

Scientific challenge:

"An inability to computationally capture the material evolution and degradation within a structural component"

X-15 (1959)

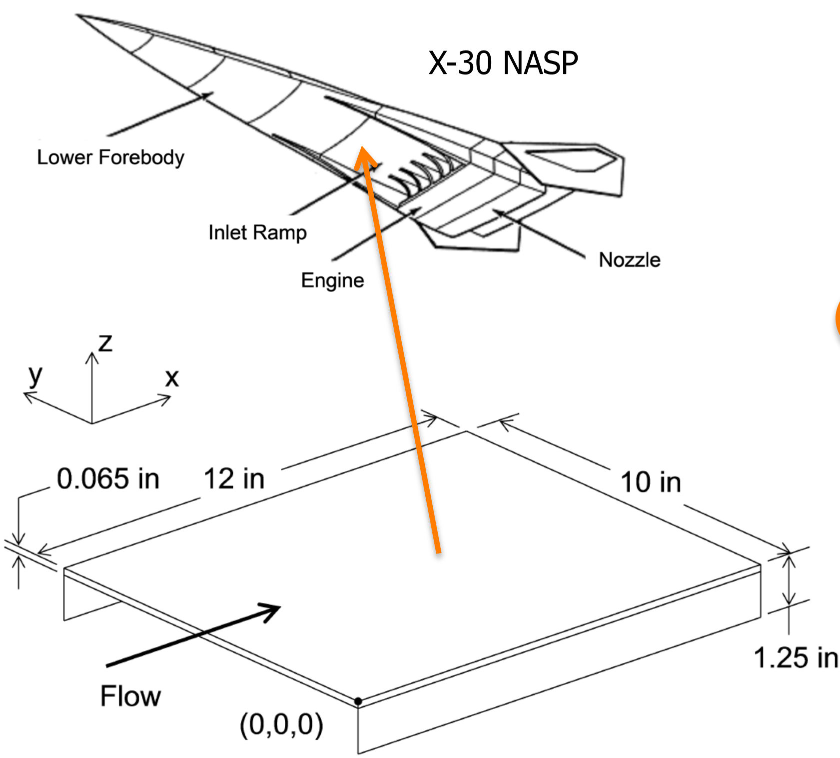


X-51a (2010)

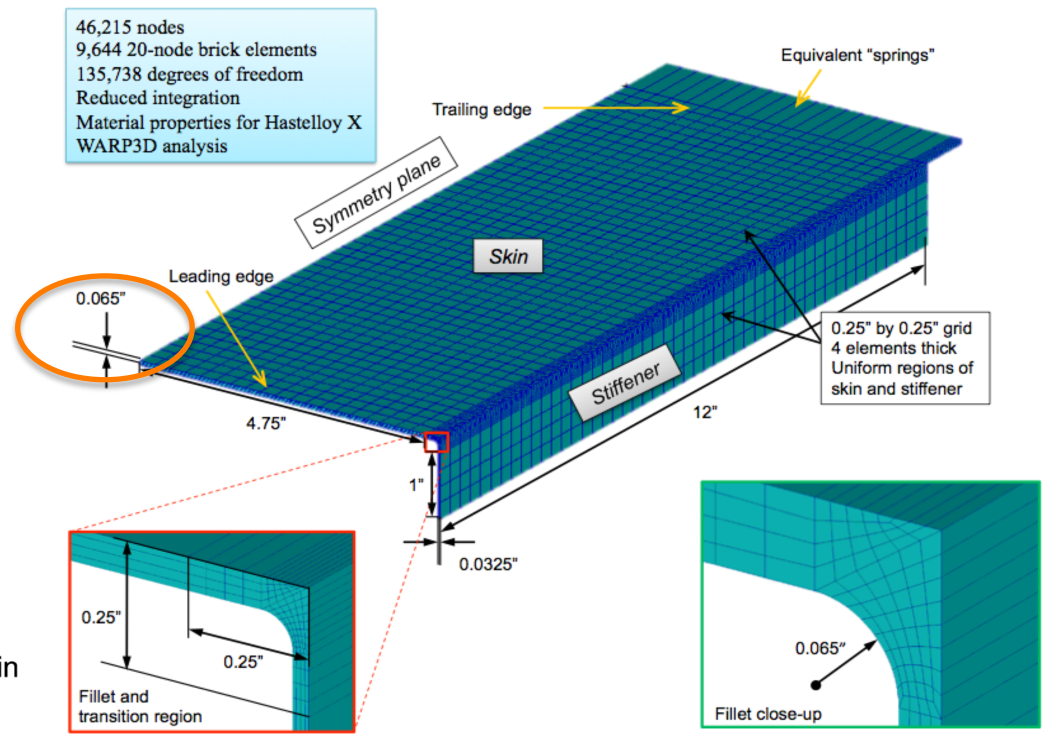
*[T. G. Eason et al., 2013]

Motivation: Multiscale Structural Analysis

- Thermo, mechanical and acoustic loads on hypersonic aircrafts lead to highly localized non-linear 3-D stress fields: Finite element models with fine meshes are required



Representative hypersonic skin panel [Sobotka et al., 2013]



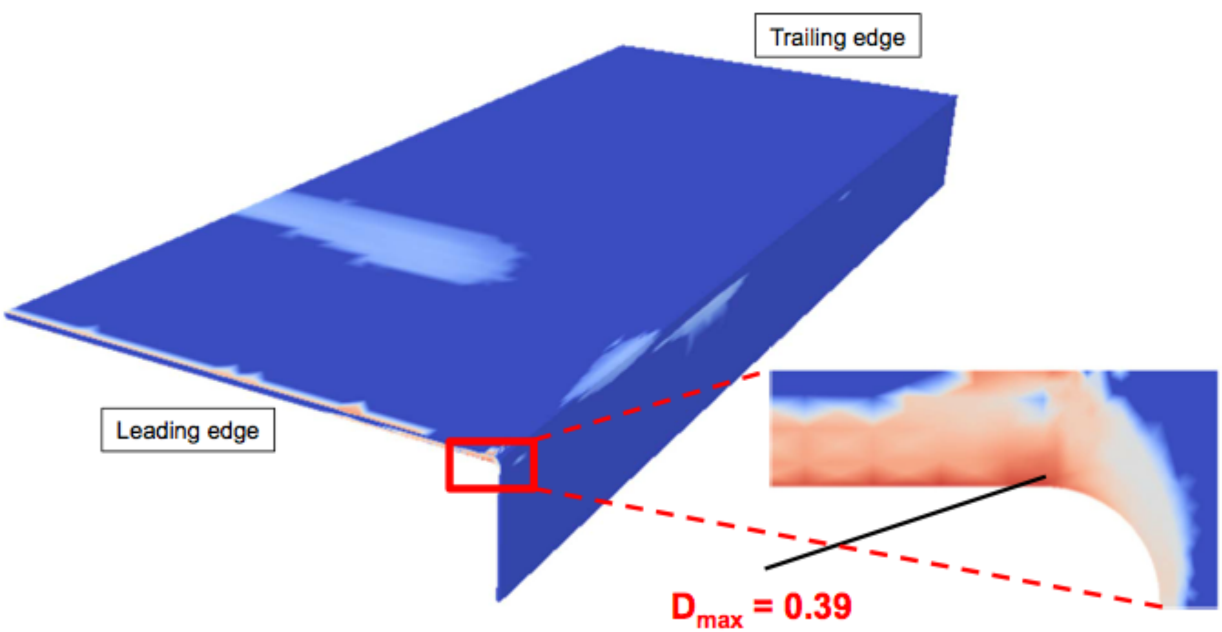
3-D FEM: Large aspect ratio of elements may lead to numerical instabilities during analysis [Sobotka et al., 2013]

Motivation: Multiscale Structural Analysis

- Highly localized non-linear 3-D effects
- Most of the structure remains linear elastic

- Q: How to efficiently capture these localized non-linear 3-D effects?
- Q: How to avoid numerical stability issues caused by aspect ratio of elements?

- Strategy: A Generalized FEM for multiscale structural analysis



[Sobotka et al., 2013]

Outline

- Motivation
- Generalized finite element methods: Basic ideas
- Bridging scales with GFEM:
  - Global-local enrichments for localized non-linearities
- Numerical examples
- Conclusions

Generalized Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

$$\mathbb{S}_{GFEM} = \mathbb{S}_{FEM} + \mathbb{S}_{ENR}$$

Low order FEM space

Enrichment space with functions related to the given problem

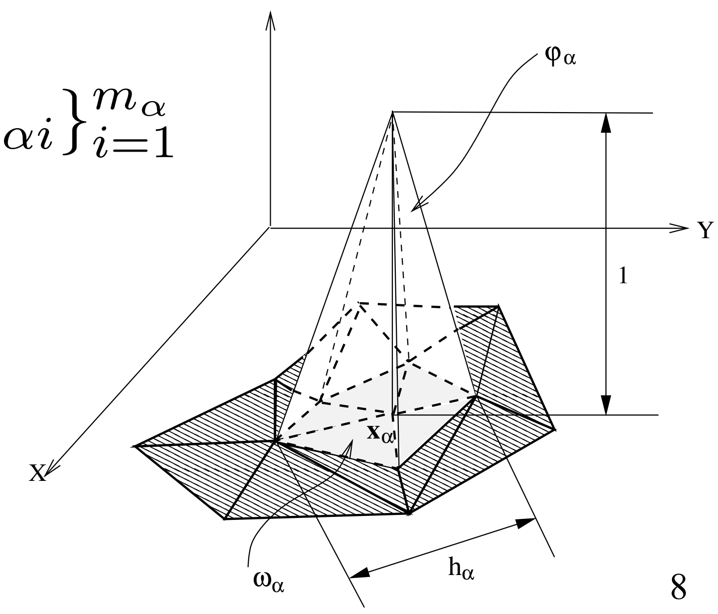
$$\mathbb{S}_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R}$$

$$\mathbb{S}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \mathrm{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}$$

$$L_{\alpha i} \in \chi_\alpha(\omega_\alpha)$$

Enrichment function

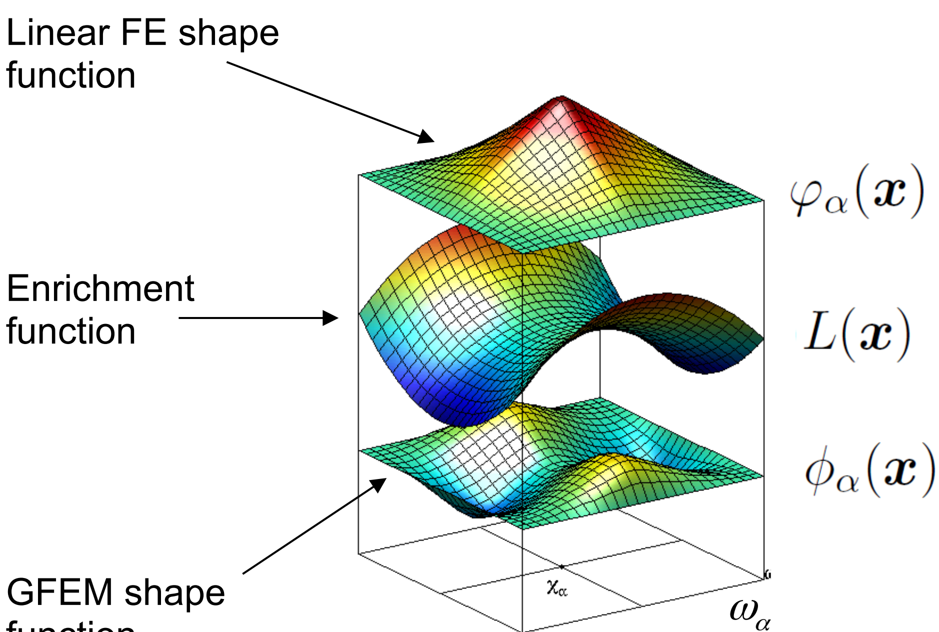
Patch space



Generalized Finite Element Method

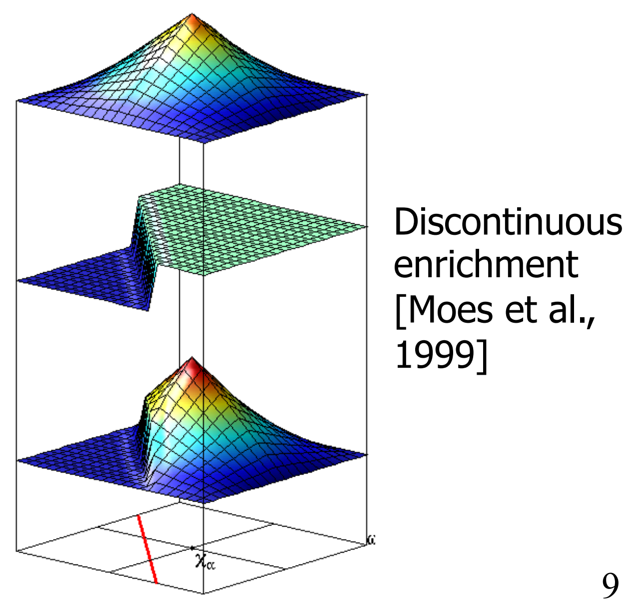
$$\mathbb{S}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}$$

$$\phi_{\alpha i}(x) = \varphi_\alpha(x) L_{\alpha i}(x) \qquad \sum_\alpha \varphi_\alpha(\boldsymbol{x}) = 1$$



[Oden, Duarte & Zienkiewicz, 1996]

• Allows construction of shape functions incorporating a-priori knowledge about solution



Multiscale Structural Analysis

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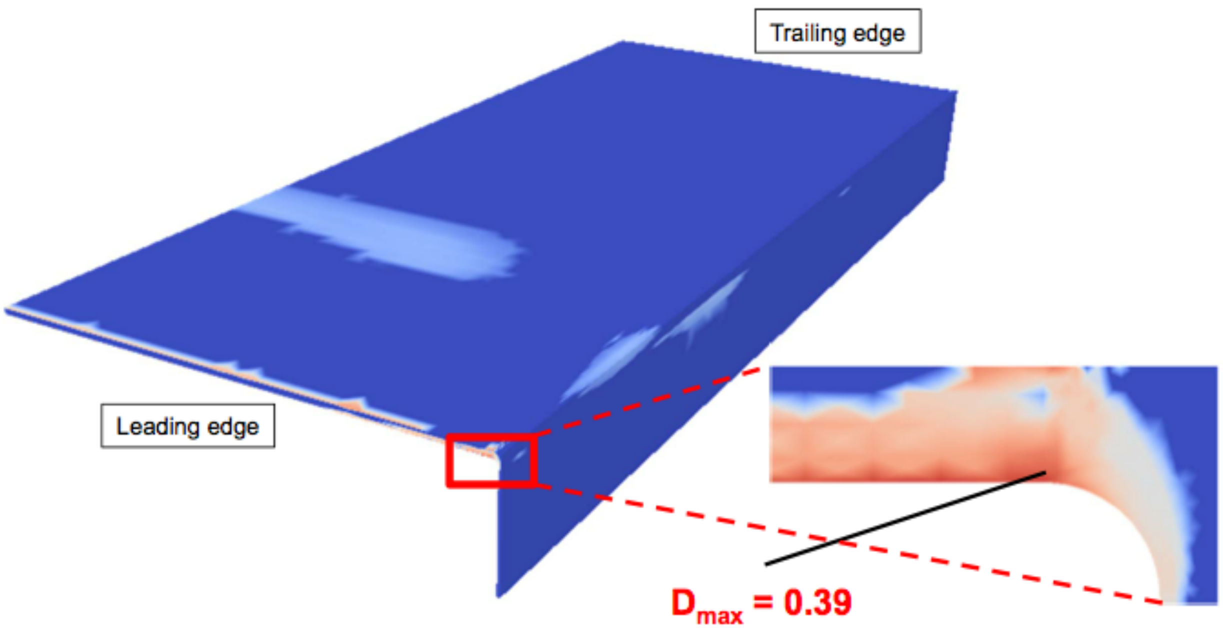
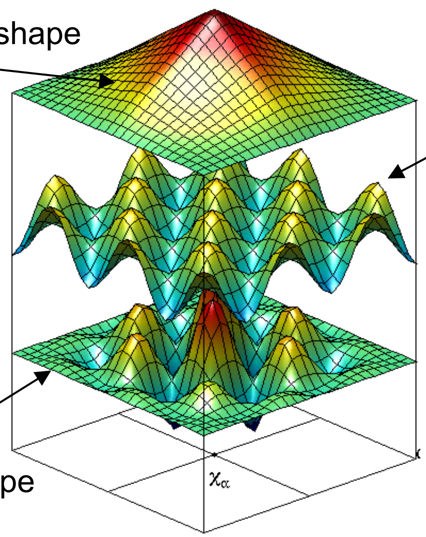


[Sobotka et al., 2013]

Bridging Scales with Global-Local Enrichment Functions*

- Enrichment functions computed from solution of local boundary value problems: Global-Local enrichment functions



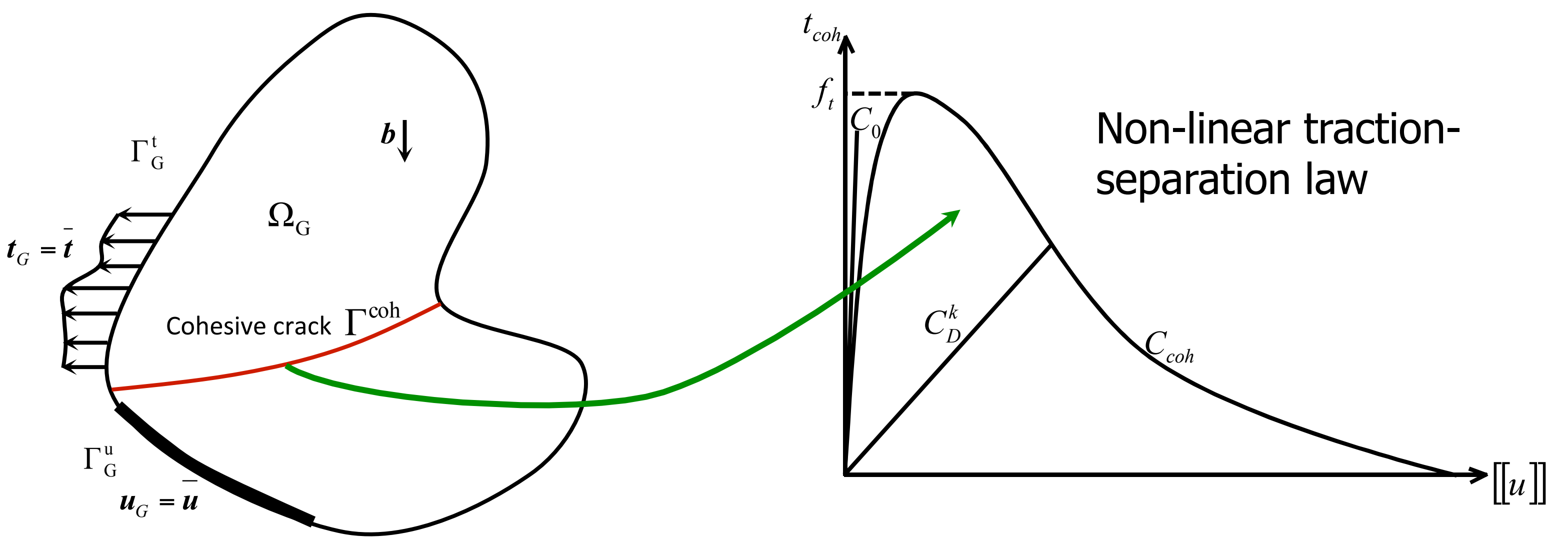
Enrichment = Numerical solutions of BVP

- **Idea:** *Use available numerical solution at a simulation step to build shape functions for next step* (quasi-static, transient, non-linear, etc.)
- Enrichment functions are produced numerically on-the-fly through a global-local analysis
- Use a *coarse* mesh enriched with Global-Local (GL) functions
- $GFEM^{gl}$ = GFEM with global-local enrichments

*[Duarte et al. 2005]

Global-Local Enrichments for Problems with Localized Non-Linearities

- **Model Problem**: Simulation of propagating cracks using cohesive fracture models



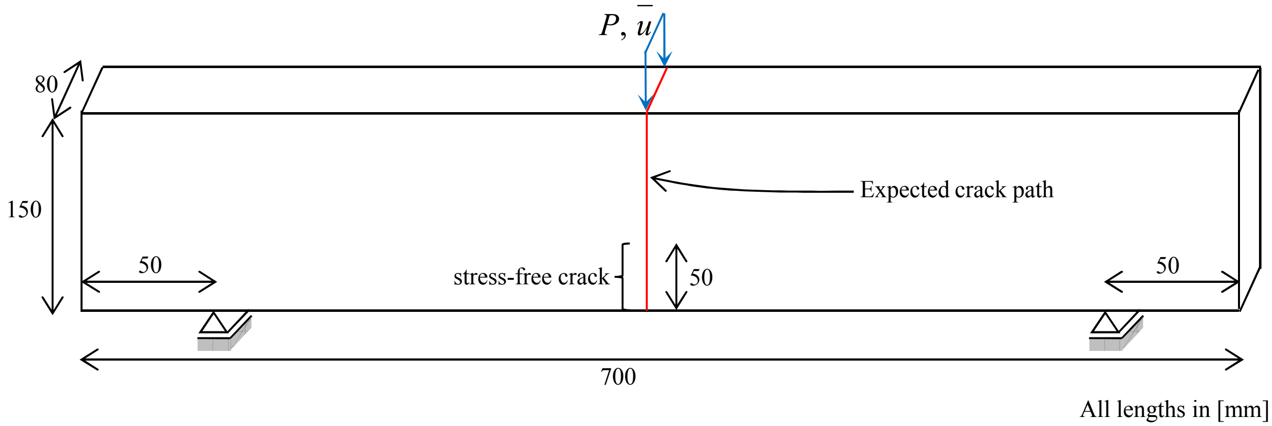
Find $\boldsymbol{u} \in H^1(\Omega_G)$, such that $\forall\ \delta\boldsymbol{u} \in H^1(\Omega_G)$

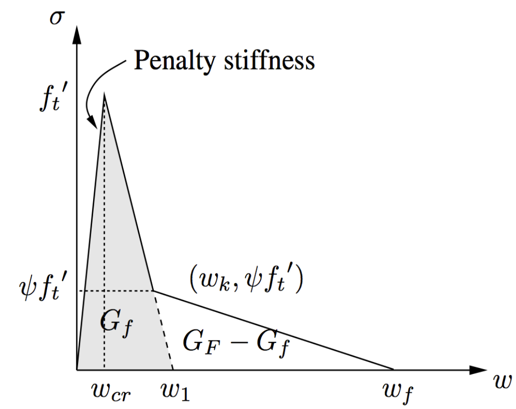
$$\int_{\Omega_G} \nabla^s(\delta\boldsymbol{u}) : \boldsymbol{\sigma}(\boldsymbol{u})\ \mathrm{dV} + \int_{\Gamma^{coh}} \delta[\![\boldsymbol{u}]\!] \cdot \boldsymbol{t}^{coh}([\![\boldsymbol{u}]\!])\ \mathrm{dS} + \eta \int_{\Gamma_G^u} \delta\boldsymbol{u} \cdot \boldsymbol{u}\ \mathrm{dS}$$

$$= \int_{\Omega_G} \delta\boldsymbol{u} \cdot \boldsymbol{b}\ \mathrm{dV} + \int_{\Gamma_G^t} \delta\boldsymbol{u} \cdot \bar{\boldsymbol{t}}\ \mathrm{dS} + \eta \int_{\Gamma_G^u} \delta\boldsymbol{u} \cdot \bar{\boldsymbol{u}}\ \mathrm{dS}$$

Global-Local Enrichments for Problems with Localized Non-Linearities

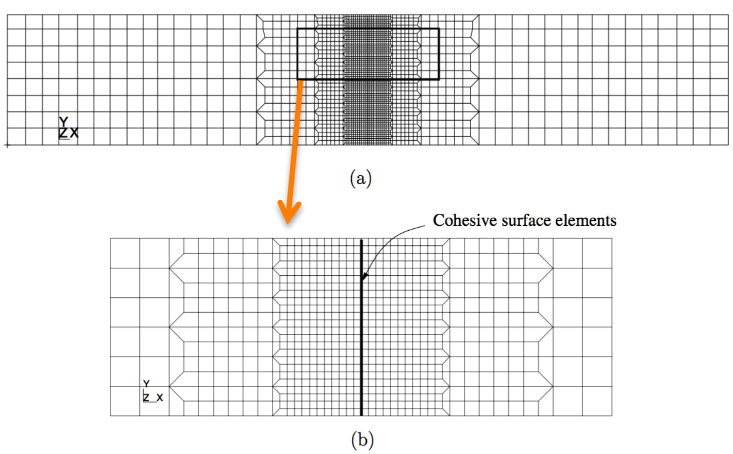
- Three-Point Bending Beam





Bi-linear intrinsic CZM [Park et al. 2008]

- Typical FEM discretization [Park et al. 2008]

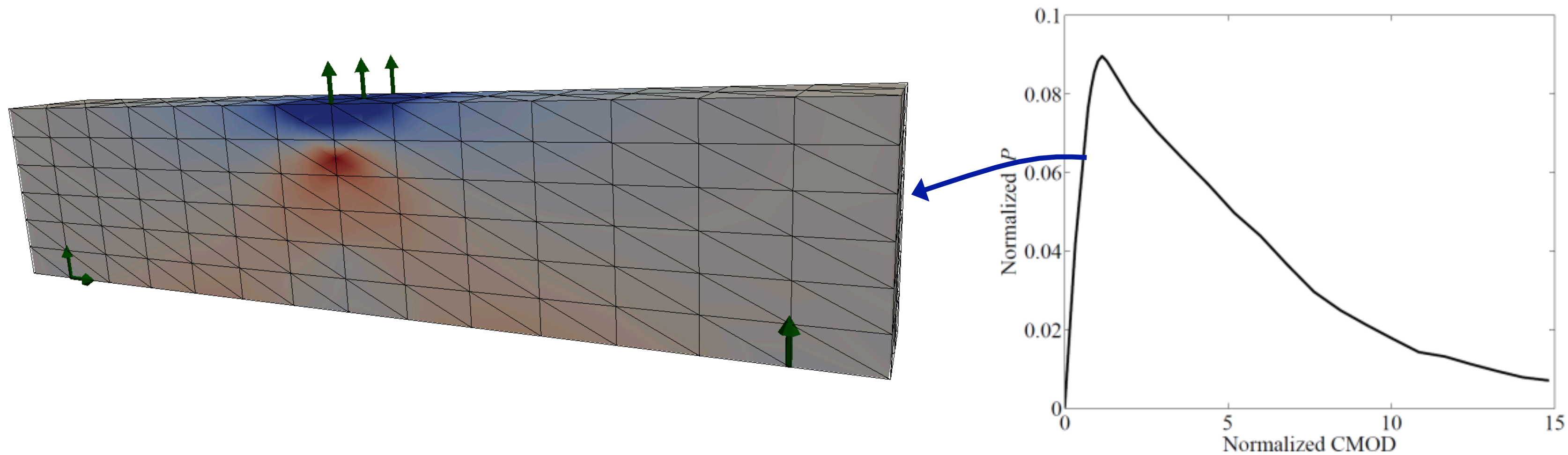


Goals:

- Solve problem on a coarse global mesh.
- Non-linear iterations at fine scales <u>only</u>.

Global-Local Enrichments for Problems with Localized Non-Linearities

Let $\boldsymbol{u}_G^n \in \mathbb{S}_G^n(\Omega)$ be a GFEM approximation of global problem at load step n. Global-local enrichments are used in the definition of $\mathbb{S}_G^n(\Omega)$.



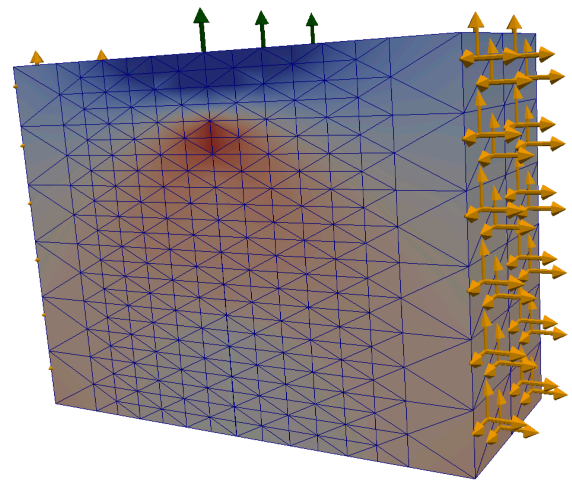
Compute a rough and cheap estimate of global solution at next load step.

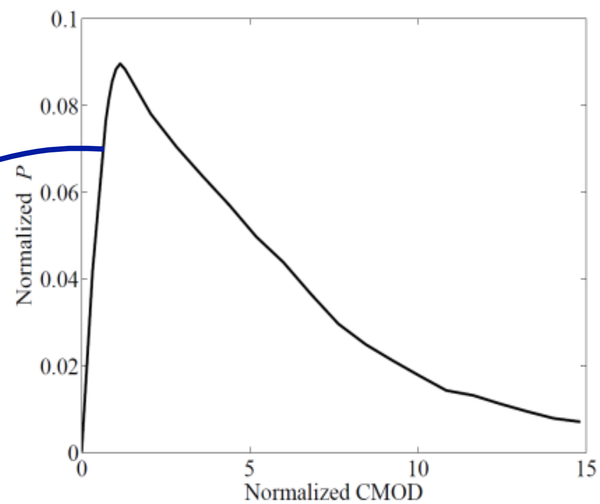
Define $\boldsymbol{u}_{G,0}^{n+1} = \frac{n+1}{n}\boldsymbol{u}_G^n$

$\boldsymbol{u}_{G,0}^{n+1}$ is used to prescribe boundary conditions for a non-linear local problem as defined next.

Global-Local Enrichments for Problems with Localized Non-Linearities

- Solve following non-linear *local* problem at load step n+1 using, e.g., *hp*-GFEM





Find $\boldsymbol{u}_L^{n+1} \in \mathbb{S}_L^{n+1}(\Omega_L)$ such that, $\forall\ \delta\boldsymbol{u}_L^{n+1} \in \mathbb{S}_L^{n+1}(\Omega_L)$

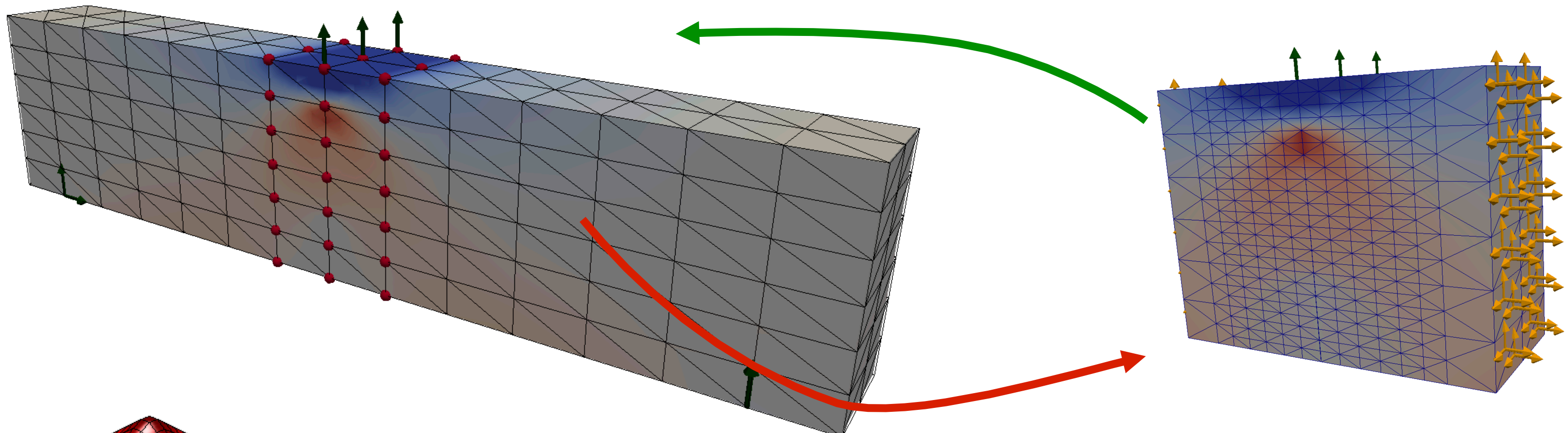
$$\int_{\Omega_\mathrm{L}} \nabla^\mathrm{s}\left(\delta\boldsymbol{u}_L^{n+1}\right) : \boldsymbol{\sigma}\left(\boldsymbol{u}_L^{n+1}\right)\ \mathrm{dV} + \int_{\Gamma^\mathrm{coh}} \delta[\![\boldsymbol{u}_L^{n+1}]\!] \cdot \boldsymbol{t}_L^\mathrm{coh}([\![\boldsymbol{u}_L^{n+1}]\!])\ \mathrm{dS} + \eta \int_{\Gamma_L \cap \Gamma_\mathrm{G}^\mathrm{u}} \delta\boldsymbol{u}_L^{n+1} \cdot \boldsymbol{u}_L^{n+1}\ \mathrm{dS}$$

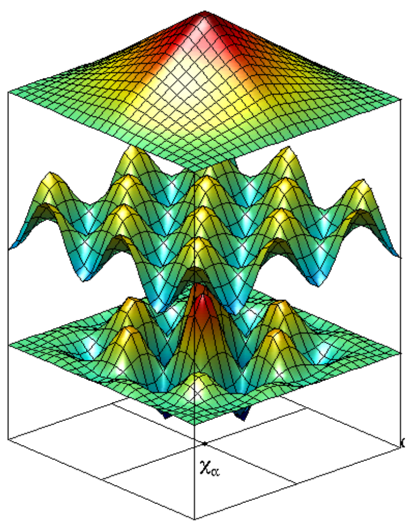
$$+ \kappa \int_{\Gamma_\mathrm{L} \backslash \left(\Gamma_\mathrm{L} \cap (\Gamma_\mathrm{G}^\mathrm{u} \cup \Gamma_\mathrm{G}^\mathrm{t})\right)} \delta\boldsymbol{u}_L^{n+1} \cdot \boldsymbol{u}_L^{n+1}\ \mathrm{dS} = \int_{\Omega_\mathrm{L}} \delta\boldsymbol{u}_L^{n+1} \cdot \boldsymbol{b}\ \mathrm{dV} + \int_{\Gamma_L \cap \Gamma_\mathrm{G}^\mathrm{t}} \delta\boldsymbol{u}_L^{n+1} \cdot \bar{\boldsymbol{t}}\ \mathrm{dS}$$

$$+ \eta \int_{\Gamma_L \cap \Gamma_\mathrm{G}^\mathrm{u}} \delta\boldsymbol{u}_L^{n+1} \cdot \bar{\boldsymbol{u}}\ \mathrm{dS} + \int_{\Gamma_\mathrm{L} \backslash \left(\Gamma_\mathrm{L} \cap (\Gamma_\mathrm{G}^\mathrm{u} \cup \Gamma_\mathrm{G}^\mathrm{t})\right)} \delta\boldsymbol{u}_L^{n+1} \cdot \left[\boldsymbol{t}_G^{n+1}\left(\boldsymbol{u}_{G,0}^{n+1}\right) + \kappa\boldsymbol{u}_{G,0}^{n+1}\right] \mathrm{dS}$$

Global-Local Enrichments for Problems with Localized Non-Linearities

- Global space enriched with non-linear local solution





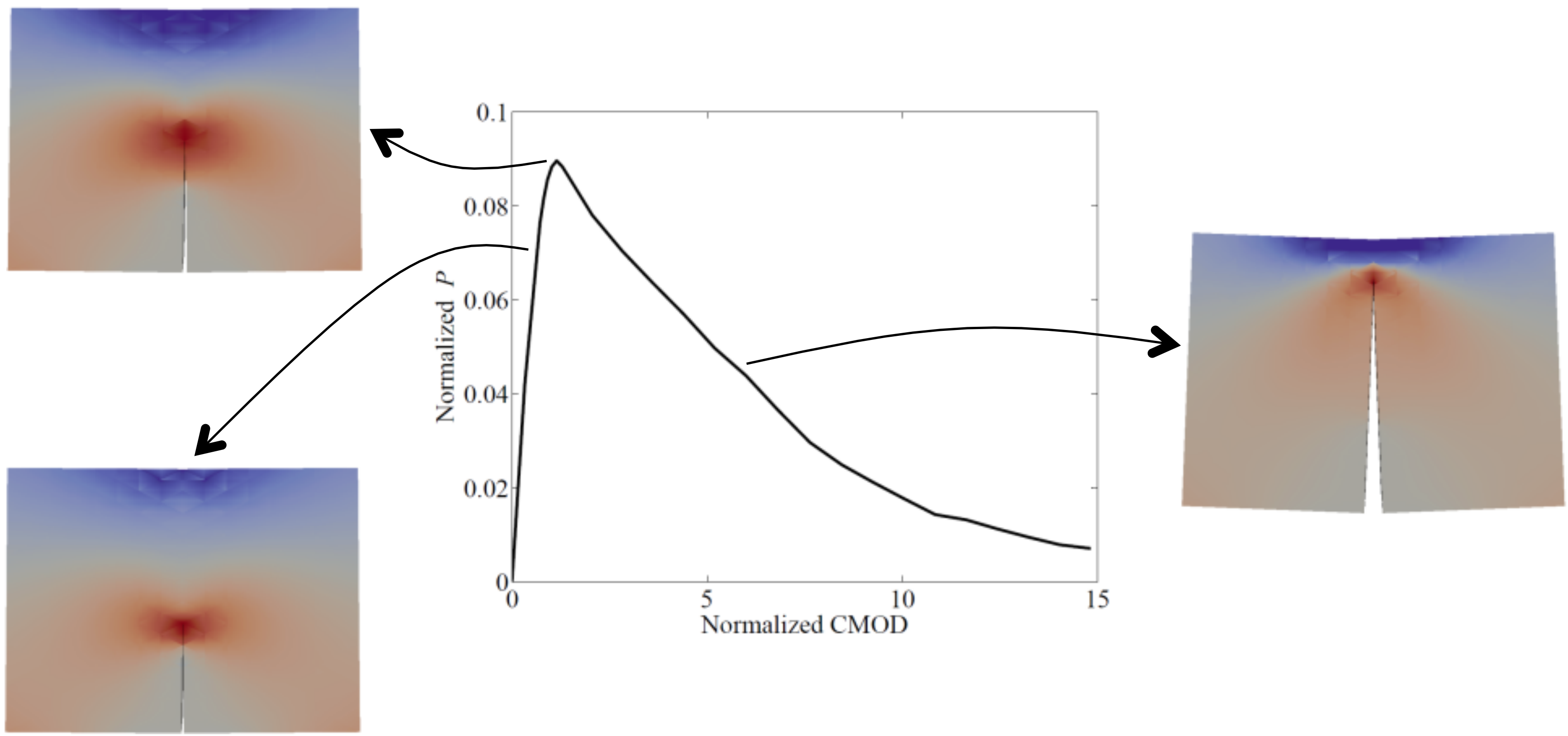
$$\phi_\alpha^{n+1}(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})\boldsymbol{u}_L^{n+1}(\boldsymbol{x})$$

$$u_G^{n+1}(\boldsymbol{x}) \in \mathbb{S}_G^{n+1}(\Omega_G) = \mathbb{S}_G^{\text{FEM}} + \left\{\varphi_\alpha \boldsymbol{u}_\alpha^{\text{gl},n+1},\ \alpha \in \mathcal{I}^{\text{gl}}\right\}$$

$$\text{where } \boldsymbol{u}_\alpha^{\text{gl},n+1}(\boldsymbol{x}) = \left\{\begin{array}{c} \underline{u}_\alpha \boldsymbol{u}_L^{n+1,<0>}(\boldsymbol{x}) \\ \underline{v}_\alpha \boldsymbol{u}_L^{n+1,<1>}(\boldsymbol{x}) \\ \underline{w}_\alpha \boldsymbol{u}_L^{n+1,<2>}(\boldsymbol{x}) \end{array}\right\},\ \underline{u}_\alpha, \underline{v}_\alpha, \underline{w}_\alpha \in \mathbb{R}$$

- Discretization spaces updated on-the-fly with global-local enrichment functions

Global-Local Enrichments for Problems with Localized Non-Linearities



- On-the-fly updating of global-local enrichment functions during the non-linear iterative solution process

Global-Local Enrichments for Problems with Localized Non-Linearities

- Global space at load step n+1

$$u_G^{n+1}(\boldsymbol{x}) \in \mathbb{S}_G^{n+1}(\Omega_G) = \mathbb{S}_G^{\mathrm{FEM}} + \mathbb{S}_G^{\mathrm{ENR,n+1}}$$

$$\text{where } \mathbb{S}_G^{\mathrm{ENR,n+1}} = \left\{\varphi_\alpha \boldsymbol{u}_\alpha^{\mathrm{gl},n+1},\ \alpha \in \mathcal{I}^{\mathrm{gl}}\right\}$$

- Discretization spaces updated on-the-fly with global-local enrichment functions
- Enrichment space is *load dependent*
- Dimension of global space does not change but its basis functions do:

Use of vector with global DOFs computed at previous load step is *not* a robust choice for the initialization of the Newton-Rhapson non-linear iterations at this load step

$\boldsymbol{d}_G^n$ and $\boldsymbol{d}_G^{n+1}$ represent coefficients of different sets of GFEM shape functions

Classical strategy: Map $\boldsymbol{u}_G^n \in \mathbb{S}_G^n(\Omega_\mathrm{G})$ into $\mathbb{S}_G^{n+1}(\Omega_\mathrm{G})$

Global-Local Enrichments for Problems with Localized Non-Linearities

- <u>Linear</u> global problem at load step n+1

$$\text{Find } \boldsymbol{u}_G^{n+1} \in \mathbb{S}_G^{n+1}(\Omega_G) \text{ such that, } \forall\ \delta\boldsymbol{u}_G^{n+1} \in \mathbb{S}_G^{n+1}(\Omega_G)$$

$$\int_{\Omega_G} \nabla^s\left(\delta\boldsymbol{u}_G^{n+1}\right) : \boldsymbol{\sigma}\left(\boldsymbol{u}_G^{n+1}\right)\, \mathrm{dV} + \int_{\Gamma^{coh}} \delta[\![\boldsymbol{u}_G^{n+1}]\!] \cdot C_D^{n+1}(\boldsymbol{u}_L^{n+1})[\![\boldsymbol{u}_G^{n+1}]\!]\, \mathrm{dS} + \eta \int_{\Gamma_G^u} \delta\boldsymbol{u}_G^{n+1} \cdot \boldsymbol{u}_G^{n+1}\, \mathrm{dS}$$

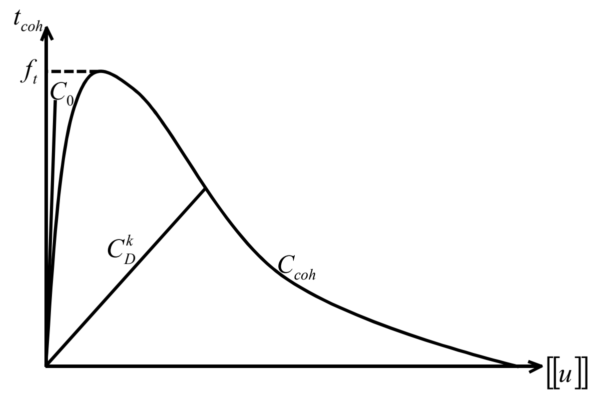
$$= \int_{\Omega_G} \delta\boldsymbol{u}_G^{n+1} \cdot \boldsymbol{b}\, \mathrm{dV} + \int_{\Gamma_G^t} \delta\boldsymbol{u}_G^{n+1} \cdot \bar{\boldsymbol{t}}\, \mathrm{dS} + \eta \int_{\Gamma_G^u} \delta\boldsymbol{u}_G^{n+1} \cdot \bar{\boldsymbol{u}}\, \mathrm{dS}$$

The cohesive secant matrix is given by

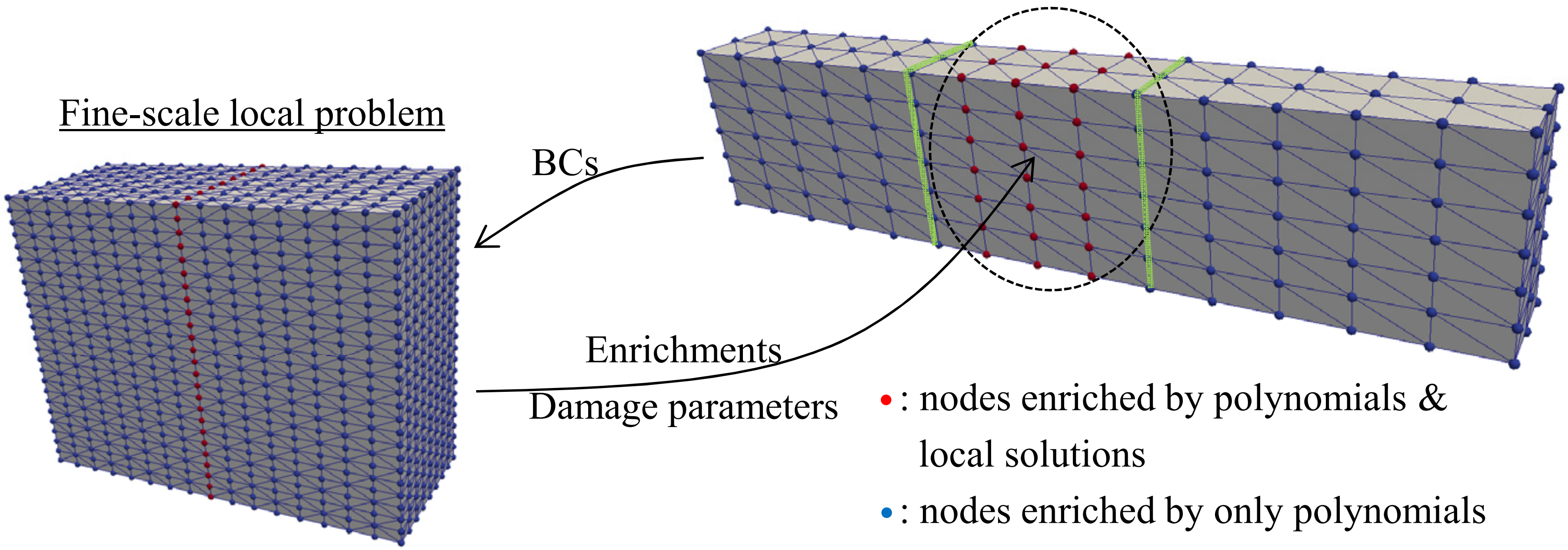
$$\mathbf{C}_D^{n+1} = \begin{bmatrix} C_D^{n+1,<m_0>} & 0 & 0 \\ 0 & C_D^{n+1,<m_1>} & 0 \\ 0 & 0 & C_D^{n+1,<m_2>} \end{bmatrix}$$

where

$$C_D^{n+1,<m_t>} = \frac{t_{coh}^{<m_t>}([\![\boldsymbol{u}_L^{n+1}]\!])}{[\![\boldsymbol{u}_L^{n+1}]\!]^{<m_t>}},\ t = 0, 1, 2$$



GFEM^{gl} Algorithm for Cohesive Fractures



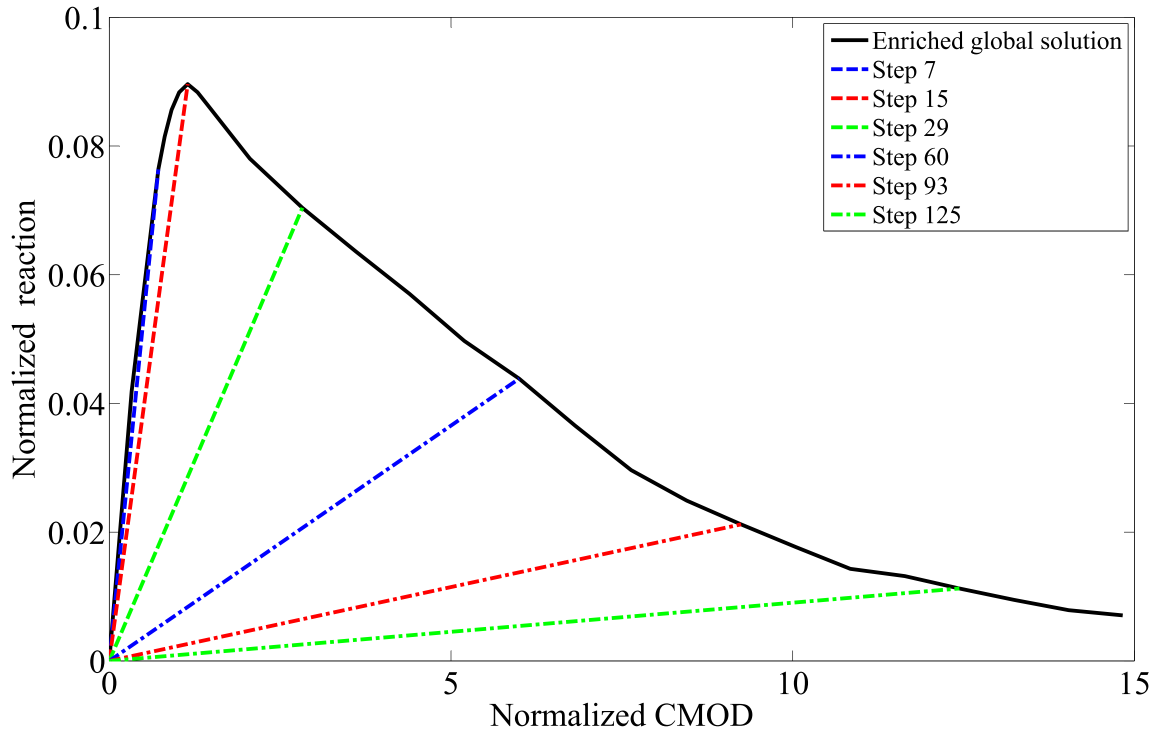
• : nodes enriched by polynomials & Heaviside functions

• : nodes enriched by only polynomials

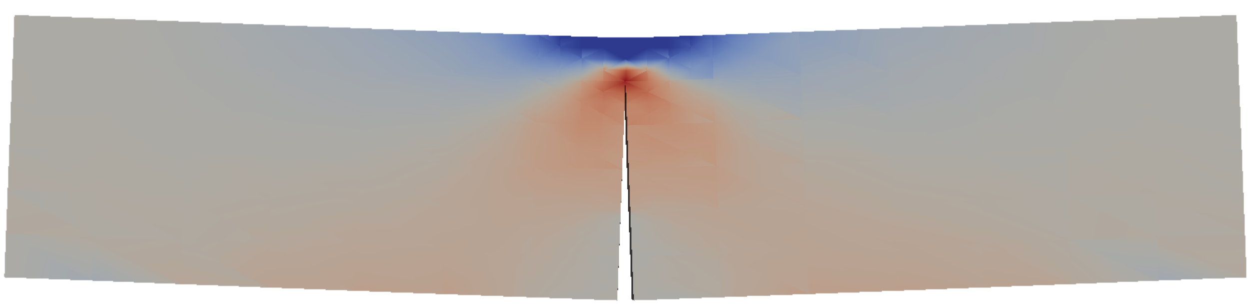
Interface between local domain boundary and global domain

- A linear global problem with secant stiffness provided by local problem is solved at every load step
- All non-linearities are handled at the local problem
- BCs for local problem can be improved through global-local iterations

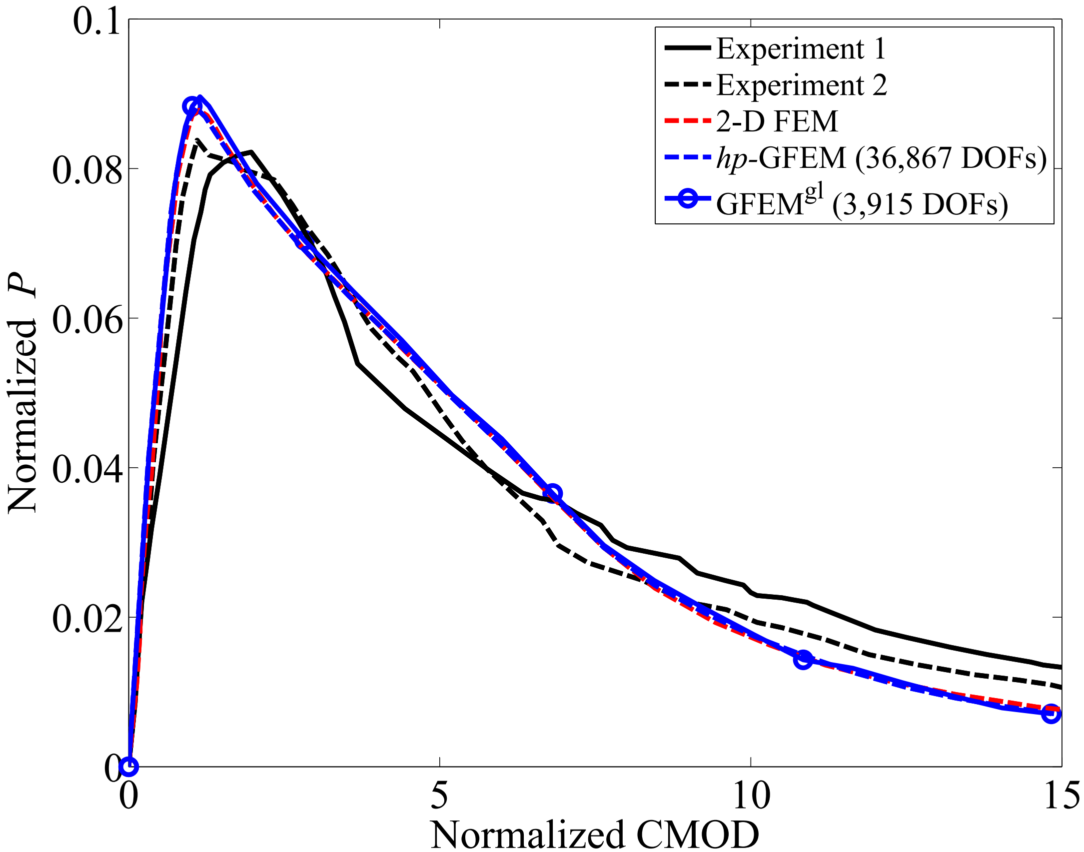
Global-Local Enrichments for Problems with Localized Non-Linearities



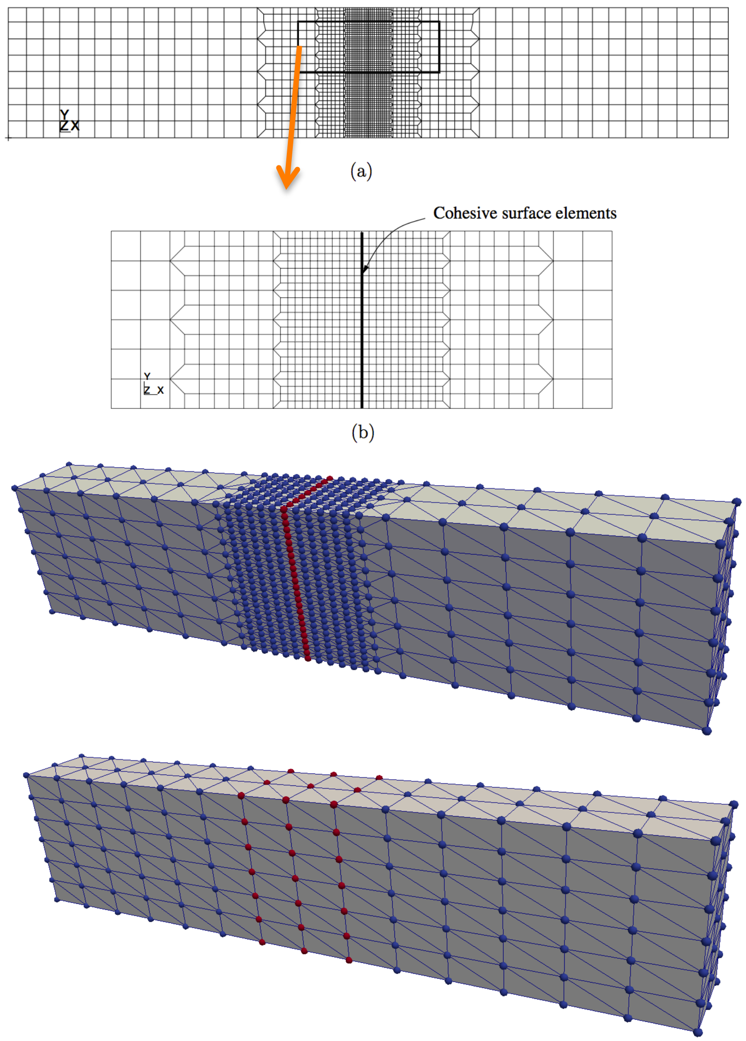
Step 60



Global-Local Enrichments for Problems with Localized Non-Linearities



Experiments by [Roesler et al., 2007],
2-D FEM results by [Park et al. 2008]



2-D FEM, *hp*-GFEM, and GFEM^{gl} meshes

Control of Non-Linear Residue of Global Solution

The residue of the original non-linear global problem is evaluated using

$$\boldsymbol{R}_G^{n+1} = \boldsymbol{f}_{G,ext}^{n+1}(\boldsymbol{d}_L^{n+1}) - \boldsymbol{f}_{G,int}^{n+1}(\boldsymbol{d}_G^{n+1}, \boldsymbol{d}_L^{n+1})$$

where the internal and external force vectors are given by

$$\boldsymbol{f}_{G,int}^{n+1}(\boldsymbol{d}_G^{n+1}, \boldsymbol{d}_L^{n+1}) = \int_{\Omega_\mathrm{G}} \overline{\boldsymbol{B}}^\mathrm{T} \boldsymbol{\sigma}(\boldsymbol{d}_G^{n+1})\,\mathrm{dV} + \int_{\Gamma^\mathrm{coh}} \overline{\boldsymbol{\phi}}^\mathrm{T} \boldsymbol{t}_{coh}(\boldsymbol{d}_G^{n+1})\,\mathrm{dS} + \eta \int_{\Gamma_\mathrm{G}^\mathrm{u}} \overline{\boldsymbol{N}}^\mathrm{T} \boldsymbol{u}_G^{n+1}(\boldsymbol{d}_G^{n+1})\,\mathrm{dS}$$

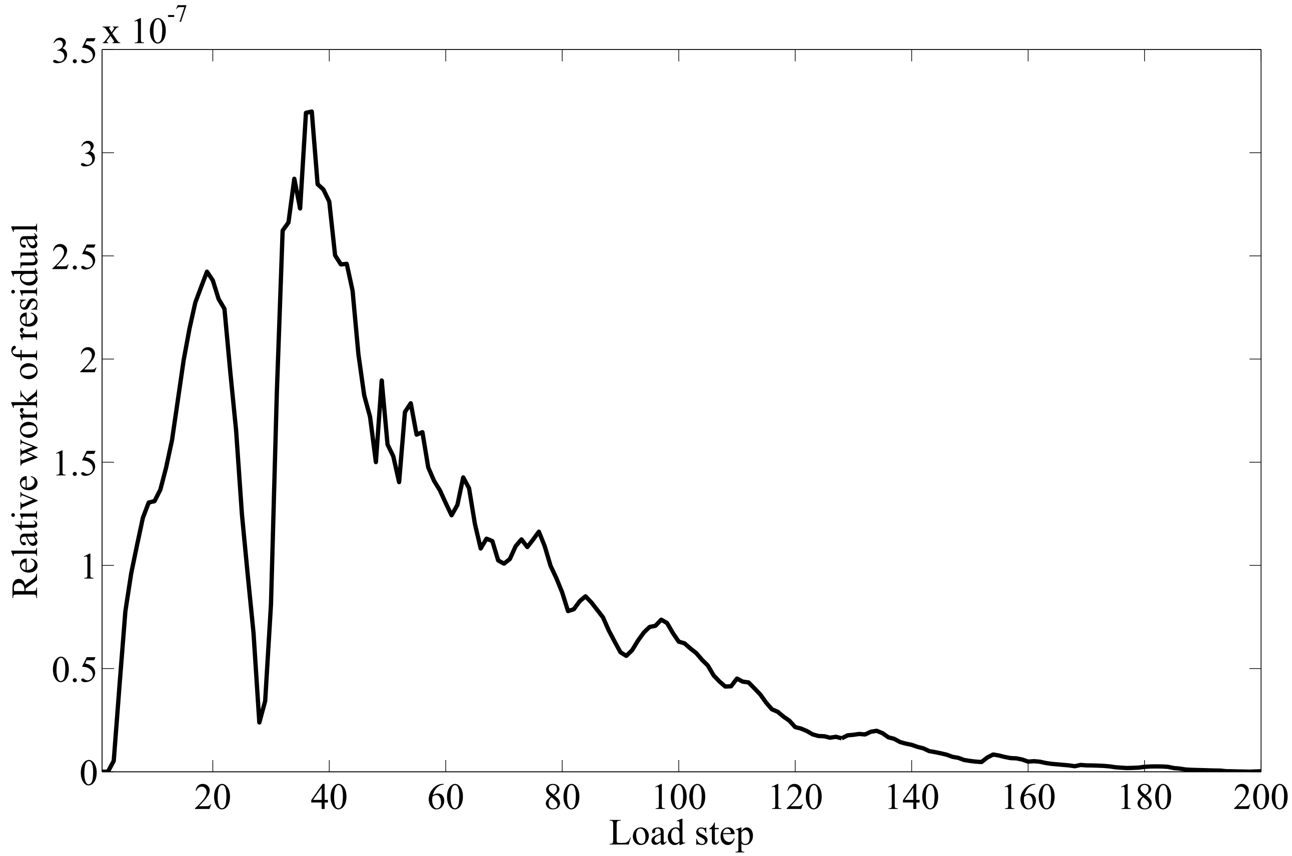
$$\boldsymbol{f}_{G,ext}^{n+1}(\boldsymbol{d}_L^{n+1}) = \int_{\Omega_\mathrm{G}} \overline{\boldsymbol{N}}^\mathrm{T} \boldsymbol{b}^{n+1}\,\mathrm{dV} + \int_{\Gamma_\mathrm{G}^\mathrm{t}} \overline{\boldsymbol{N}}^\mathrm{T} \bar{\boldsymbol{t}}^{n+1}\,\mathrm{dS} + \eta \int_{\Gamma_\mathrm{G}^\mathrm{u}} \overline{\boldsymbol{N}}^\mathrm{T} \bar{\boldsymbol{u}}^{n+1}\,\mathrm{dS}$$

The magnitude of residue relative to external forces is measured using

$$W_G^{n+1} = \frac{{\boldsymbol{R}_G^{n+1}}^\mathrm{T} \boldsymbol{d}_G^{n+1}}{{\boldsymbol{f}_{G,ext}^{n+1}}^\mathrm{T} \boldsymbol{d}_G^{n+1}}$$

Control of Non-Linear Residue of Global Solution

- Non-linear residue of global solution computed with secant stiffness



- Residue is maximum at load step 37 while limit point is reached at load step 28
- <u>One</u> Newton-Raphason iteration reduces residue to ~10^{-11} at all load steps

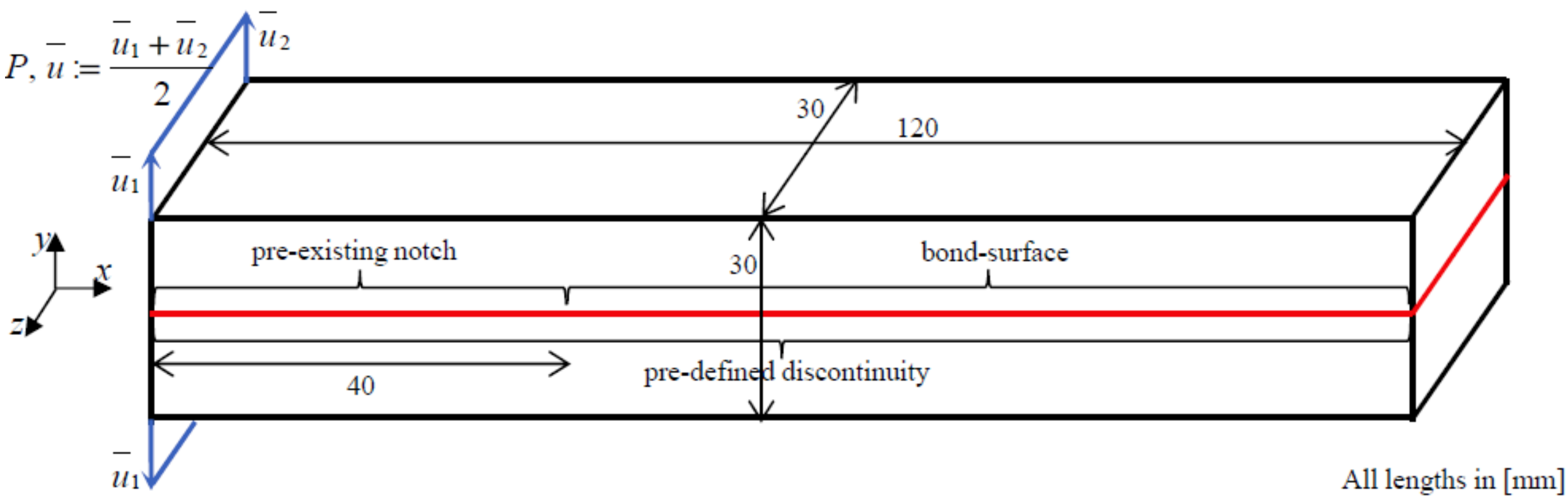
Control of Non-Linear Residue of Global Solution

If the residue $W_G^{n+1} = \dfrac{{\boldsymbol{R}_G^{n+1}}^{\mathrm{T}} \boldsymbol{d}_G^{n+1}}{{\boldsymbol{f}_{G,ext}^{n+1}}^{\mathrm{T}} \boldsymbol{d}_G^{n+1}}$ is above acceptable tolerance:

- Solve non-linear global problem using Newton-Rhapson iterations with a cohesive tangent matrix instead of secant stiffness
- The solution of the global problem computed using secant stiffness provides a robust initialization of the iterative process

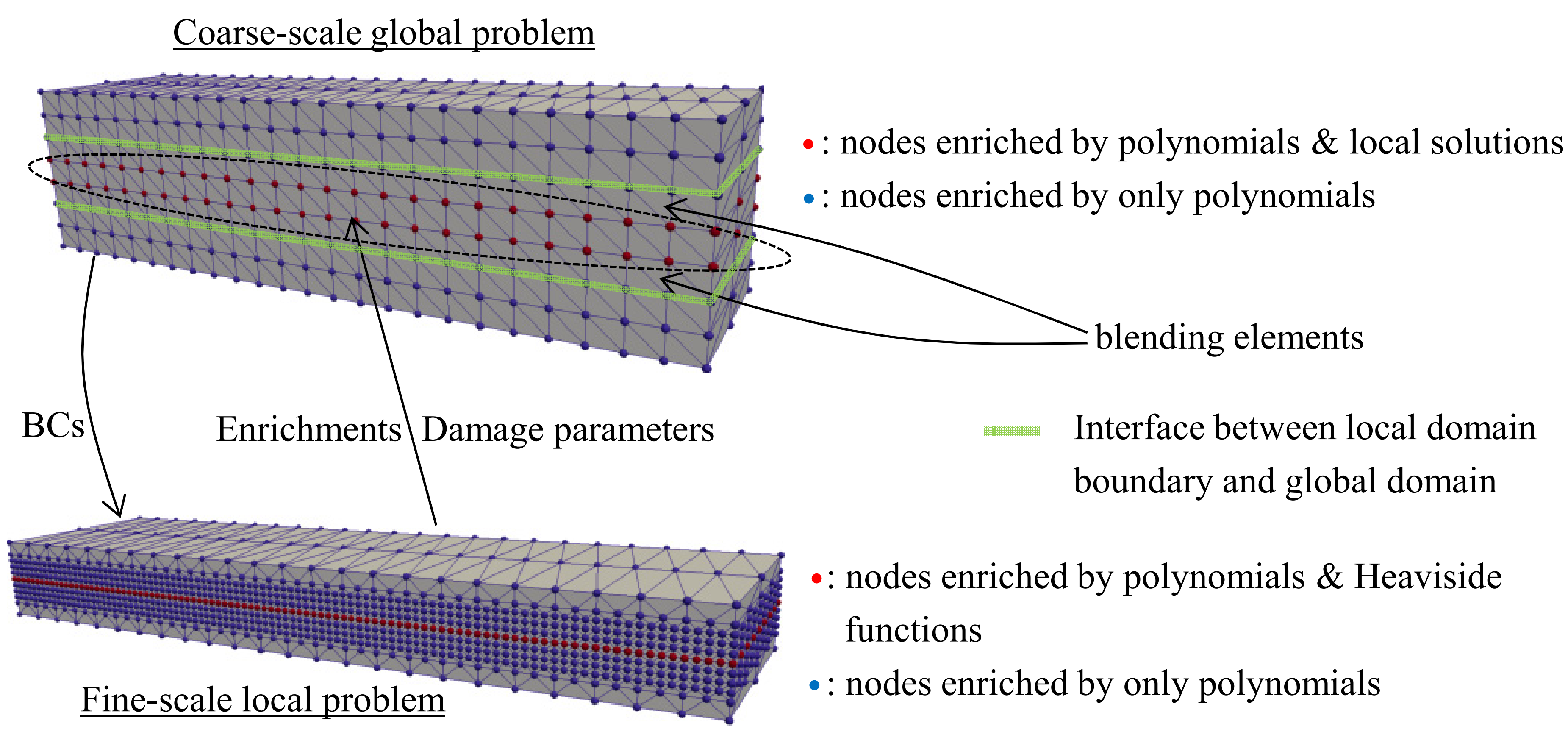
Delamination Test: Problem Setup

Bonded double cantilever beam (DCB) specimen



- $\bar{u}_1 = \bar{u}_2$: mode I, $\bar{u}_1 = 2\bar{u}_2$: mixed mode
- Isotropic linear elasticity in the bulk
- Linear and Paulino-Park-Roesler (PPR) cohesive models adopted

GFEM^{gl} Algorithm for Cohesive Fractures

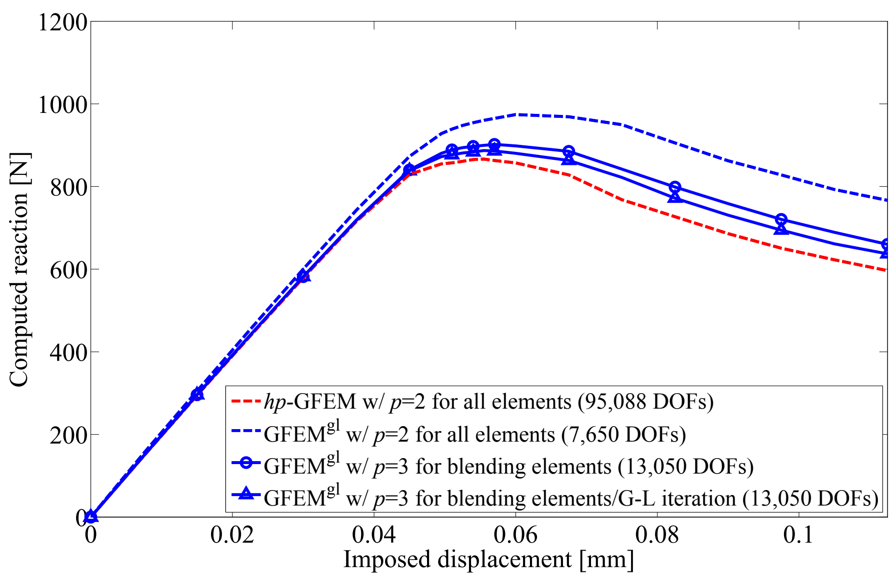


Delamination Test

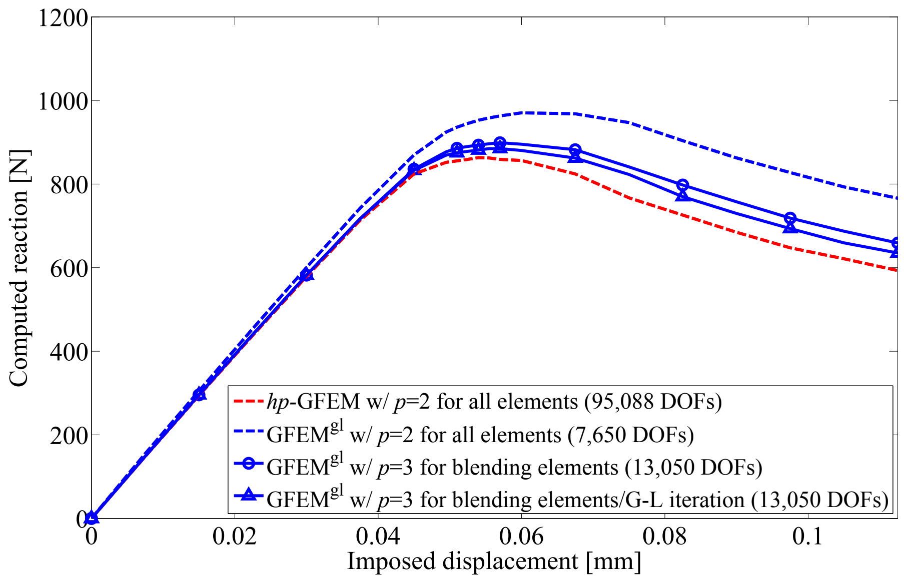
- Solution for mixed mode case, linear cohesive law
- Stress component normal to crack is shown

Delamination Test

Computed reaction versus imposed average displacement: Mixed mode case



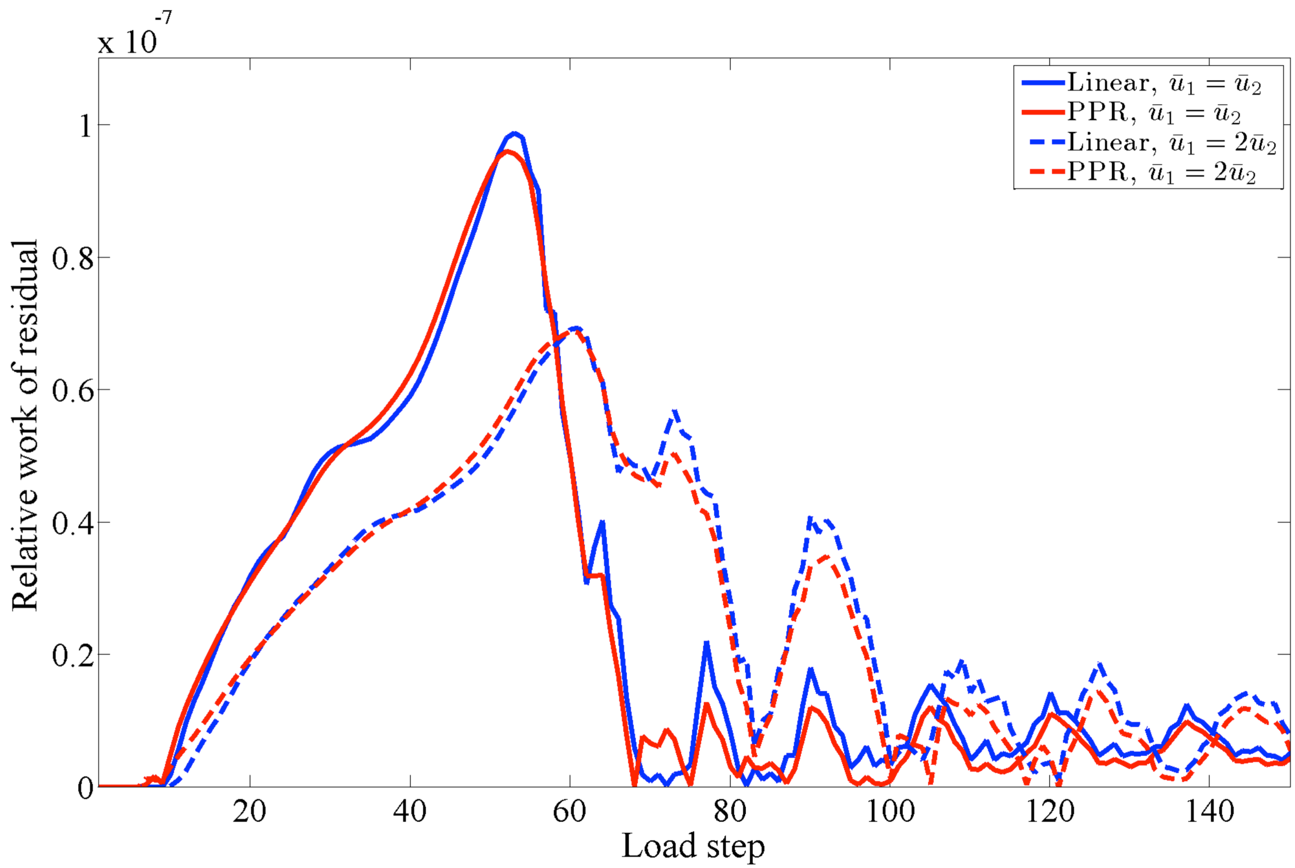
Linear cohesive law



PPR cohesive law

Delamination Test

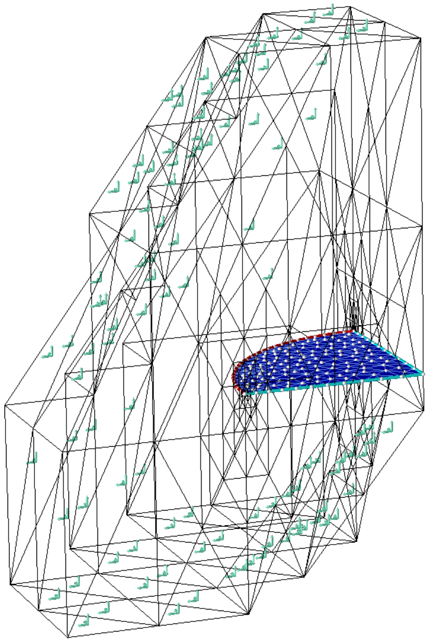
- Non-linear residue of global solution computed with secant stiffness



- Load steps corresponding to the limit point are *not* the same as those with maximum residue
- <u>One</u> Newton-Raphason iteration reduces residue to ~10^{-11} at all load steps

Conclusions and Outlook

- Proposed GFEMgl enables effective resolution of highly localized non-linearities on coarse, structural-scale finite element meshes
- Neither global nor local meshes need to fit crack surface
- Method can be used with virtually any cohesive model and does not require a-priori knowledge about the exact solution of the problem
- Non-linear Newton-Rhapson iterations are in general performed only at local problems used in the computation of enrichment functions
- Solutions with comparable accuracy to those provided by adaptive GFEM are obtained with significantly fewer degrees of freedom at the global problem
- Amenable to non-intrusive integration with existing FEA software: Transition to Labs and Industries
- Extensions to fluid-driven cracks (hydraulic fractures) are underway



Questions?

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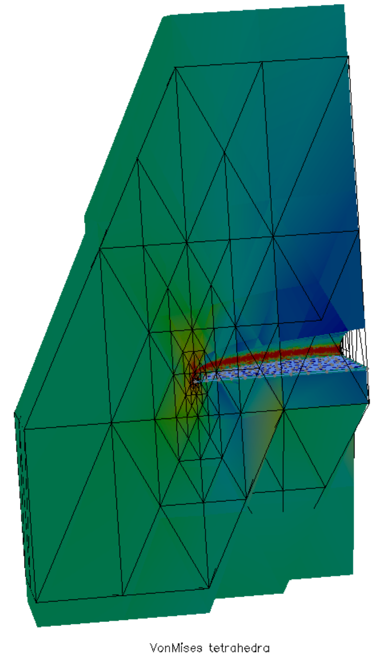

Acknowledgments