Recent Developments in the Generalized Finite Element Method and Applications in 3-D Fracture Propagation and Coalescence

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Understanding crack coalescence is of great importance in many applications.

Coalescence of fatigue micro-cracks

Cluster of hydraulic fractures propagating from a horizontal well

Reflective crack in asphalt overlay
Modeling 3-D Fractures: Limitations of Standard FEM

- It is not “just” fitting the 3-D evolving fracture
- FEM meshes must satisfy special requirements for acceptable accuracy
Limitations of Standard FEM

- Difficulties arise if fracture front is close to complex geometrical features
- Fracture surfaces with sharp turns
- Coalescence of fractures

- Not possible in general to automatically create structured meshes along both fracture fronts when they are in close proximity

- Even with these crafted meshes and quarter-point elements, convergence rate of std FEM is slow (controlled by singularity at fracture front)

- **Strategy**: Generalized FEM
Outline

- Motivation and limitations of standard finite element methods
- Basic ideas of GFEM
- GFEM for 3-D fractures
- Application
  - Propagation and coalescence of hydraulic fractures
- Conclusions and outlook
Early Works on Generalized FEMs

- Babuska, Caloz and Osborn, 1994 (Special FEM).
- Duarte and Oden, 1995 (Hp Clouds).
- Babuska and Melenk, 1995 (PUFEM).
- Oden, Duarte and Zienkiewicz, 1996 (Hp Clouds/GFEM).
- Duarte, Babuska and Oden, 1998 (GFEM).
- Belytschko et al., 1999 (Extended FEM).
- Strouboulis, Babuska and Copps, 2000 (GFEM).

- Basic idea:
  - Use a partition of unity to build Finite Element shape functions

- Review paper

“The XFEM and GFEM are basically identical methods: the name generalized finite element method was adopted by the Texas school in 1995–1996 and the name extended finite element method was coined by the Northwestern school in 1999.”
Generalized Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

\[ \mathbb{S}_{GFEM} = \mathbb{S}_{FEM} + \mathbb{S}_{ENR} \]

Low order FEM space \( \mathbb{S}_{FEM} \)

Enrichment space with functions related to the given problem \( \mathbb{S}_{ENR} \)

\[ \mathbb{S}_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R} \]

\[ \mathbb{S}_{ENR} = \sum_{\alpha \in I_{h}^{c} \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha} \]

\[ L_{\alpha i} \in \chi_\alpha(\omega_\alpha) \]

Enrichment function \( L_{\alpha i} \)

Patch space \( \chi_\alpha \)

\( \varphi_\alpha \)
Generalized Finite Element Method

\[
S_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \text{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}
\]

\[
\phi_{\alpha i}(x) = \varphi_\alpha(x) L_{\alpha i}(x) \quad \sum_\alpha \varphi_\alpha(x) = 1
\]

- Allows construction of shape functions incorporating a-priori knowledge about solution

[Oden, Duarte & Zienkiewicz, 1996]
\[ S_{GFEM}(\Omega) = \left\{ \mathbf{u}^{hp} = \sum_{\alpha \in I_h} \varphi_\alpha(\mathbf{x}) \left[ \hat{\mathbf{u}}_\alpha(\mathbf{x}) + \mathcal{H}\tilde{\mathbf{u}}_\alpha(\mathbf{x}) + \ddot{\mathbf{u}}_\alpha(\mathbf{x}) \right] \right\} \]

\[ L^{\xi}_\alpha(r, \theta) = \sqrt{r} \left[ (\kappa + \frac{1}{2}) \sin \frac{\theta}{2} - \frac{1}{2} \sin \frac{3\theta}{2} \right] \]

\[ L^{\eta}_\alpha(r, \theta) = \sqrt{r} \left[ (\kappa - \frac{1}{2}) \cos \frac{\theta}{2} - \frac{1}{2} \cos \frac{3\theta}{2} \right] \]

\[ L^{\zeta}_\alpha(r, \theta) = \sqrt{r} \sin \frac{\theta}{2} \]
Modeling Fractures with the GFEM

- Fractures are modeled via enrichment functions, not the FEM mesh.
- Mesh refinement still required for acceptable accuracy.

\[ \text{hp-GFEM} \]

\[ \text{Von Mises stress} \]

[Duarte et al., International Journal Numerical Methods in Engineering, 2007]
3D Fracture Surface Representation

- High-fidelity explicit representation of fracture surfaces [Duarte et al., 2001, 2009]

- Coalescence of fractures [Garzon et al., 2014]
Conditioning of GFEM Approximations

- The conditioning of the G/XFEM stiffness matrix, $K_{GFEM}$, can be much worse than that of the standard FEM, $K_{FEM}$

$$\overline{\mathcal{R}}(K_{GFEM}) = \mathcal{O}(h^{-4})$$

while

$$\overline{\mathcal{R}}(K_{FEM}) = \mathcal{O}(h^{-2})$$

where $\overline{\mathcal{R}}(.)$ is the scaled condition number.
The SGFEM involves simple local modifications of enrichments used in the GFEM

$$\tilde{L}_{\alpha j}(x) = L_{\alpha j}(x) - I_{\omega_{\alpha}}(L_{\alpha j})(x)$$

where $I_{\omega_{\alpha}}(L_{\alpha j})$ is the piecewise linear FE interpolant of $L_{\alpha j}$ on the patch $\omega_{\alpha}$

[Babuška & Banerjee CMAME 2012; Gupta, Duarte, Babuška & Banerjee CMAME, 2013]
SGFEM: Stable Generalized FEM

Modification of enrichment functions

\[ \tilde{L}_{\alpha i}(\mathbf{x}) = L_{\alpha i}(\mathbf{x}) - I_{\omega \alpha}(L_{\alpha i})(\mathbf{x}) \]

\[ \tilde{\phi}_{\alpha i}(\mathbf{x}) = \phi_{\alpha}(\mathbf{x})\tilde{L}_{\alpha i}(\mathbf{x}) \]
SGFEM: Stable Generalized FEM

Conditioning of GFEM/XFEM stiffness matrix $\mathcal{O}(h^{-4})$

Conditioning of SGFEM and FEM stiffness matrix $\mathcal{O}(h^{-2})$

[Gupta, Duarte, Babuska & Banerjee CMAME, 2013]
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What is Hydraulic Fracturing?


Video
Hydraulic Fracturing Simulation

Current Focus: 3-D effects not captured by available simulators

- Initial stages of fracture propagation: Fracture re-orientation, interaction and coalescence

Strategy: Generalized Finite Element Methods
Selection of Enrichment Functions: Hydraulic Fracturing Regimes

- Fracture propagation is governed by
  - two competing energy dissipation mechanisms: Viscous flow and fracturing process;
  - two competing storage mechanisms: In the fracture and in the porous matrix

\[ \mathcal{K} = \frac{4K_{Ic}}{\sqrt{\pi}} \left( \frac{1}{3Q_0 E'^3 \mu} \right)^{1/4} \]

\[ C = 2C_L \left( \frac{E' t}{12 \mu Q_0^3} \right)^{1/6} \]

**Current Focus: Storage-toughness dominated regime**

- Low permeability reservoirs: Neglect flow of hydraulic fluid across fracture faces:
  - Storage dominated regime
- High confining stress (no fluid lag) and low viscosity fluid (water):
  - Near constant fluid pressure in fracture; Toughness dominated regime
- Brittle elastic material

*[Carrier & Granet, EFM, 2013]*
Selection of Enrichment Functions: Hydraulic Fracturing Regimes

Enrichments for toughness-dominated regime:

\[ S_{GFEM}(\Omega) = \left\{ u^{hp} = \sum_{\alpha \in I_h} \varphi_\alpha(x) \right\} \]

\[ \begin{align*}
\hat{u}_\alpha(x) & \quad \text{polynomial} \\
\mathcal{H}\tilde{u}_\alpha(x) & \quad \text{discontinuous} \\
\check{u}_\alpha(x) & \quad \text{singular}
\end{align*} \]

\[ L^\xi_{\alpha_1}(r, \theta) = \sqrt{r} \left[ (\kappa - \frac{1}{2}) \cos \frac{\theta}{2} - \frac{1}{2} \cos \frac{3\theta}{2} \right] \]

\[ L^\eta_{\alpha_1}(r, \theta) = \sqrt{r} \left[ (\kappa + \frac{1}{2}) \sin \frac{\theta}{2} - \frac{1}{2} \sin \frac{3\theta}{2} \right] \]

\[ L^\zeta_{\alpha_1}(r, \theta) = \sqrt{r} \sin \frac{\theta}{2} \]

[Duarte & Oden 1996]

Valid for toughness-dominated problems

patch \( \omega_\alpha \)
Weak Form at Propagation Step $k$

Find $u^k \in H^1(\Omega)$, such that $\forall \, v^k \in H^1(\Omega)$

\[
\int_{\Omega} \sigma(u^k) : \varepsilon(v^k) \, d\Omega = \int_{\Omega} b \cdot v^k \, d\Omega + \int_{\partial\Omega} t \cdot v^k \, d\Gamma + \int_{\Gamma^k_c} t^{k+}_c \cdot [v^k] \, d\Gamma
\]

where $[v^k]$ is the virtual displacement jump across the crack surface $\Gamma^k_c$ at propagation step $k$ and

\[
t^{k+}_c = -p^k n^{k+} = p^k n^{k-}
\]

- Pore fluid flow not considered
- Toughness-dominated problem: Pressure on fracture is assumed constant
Verification: Propagation of Circular Fracture*

Geometrical and Computational fracture surface loaded with fluid pressure $p$

*Gupta & Duarte, 2014*
Verification: Propagation of Circular Fracture

Goal: Find critical pressure

Adopt [Bourdin et al. 2012]:

\[ p_c(a) = \left( \frac{E^* G_c \pi}{4a} \right)^{1/2} \]

\[ E^* = 1 \]
\[ G_c = 1.91 \times 10^{-9} \]
\[ a = 0.5 \]
\[ p_c(0.5) = 5.477 \times 10^{-5} \]
Propagation of Circular Fracture

GFEM Model

\[
\begin{align*}
  h_{\text{min}}/a &= 0.016 \\
  h_{\text{max}}/a &= 0.027 \\
  p\text{-order} &= 2 \\
  N &= 215376 \text{ dofs} \\
  T &= 5.25 \text{ min}
\end{align*}
\]

Critical pressure

\[
p_c^h(a) = \frac{K_c}{K(a)} p
\]

\[
p_{\text{c}}^h(0.5) = 5.415 \times 10^{-5}
\]

\[
e_r(p_c) = 1.15\%
\]
Propagation of Circular Fracture

Incrementally advance fracture and repeat previous approach
Application: Fracture Re-Orientaion*

- Fracture starts in a direction not perpendicular to minimum *in-situ* stress
- Misalignment of fracture and confining *in-situ* stresses

\[
\begin{align*}
    a &= 10 \text{ m} \\
    b &= 5 \text{ m} \\
    h &= 15 \text{ m} \\
    p &= 3.5 \text{ MPa}
\end{align*}
\]

*[Rungamornrat et al., 2005; Gupta & Duarte, 2014]*
Fracture Re-Orientation

- 2b
- 2a
- 45°
Fracture Re-Orientation
Fracture Re-Orientation: Step 20
Fracture Re-Orientation: Adaptive Mesh

- Adaptive refinement along fracture front
- Sharp features are preserved
- High fidelity of fracture surface, regardless of computational mesh
Typical Hydraulic Fracturing

FRACTURE DEVELOPMENT AS FUNCTION OF WELLBORE ORIENTATION

[Z. Rahim et al., 2012]
Longitudinal Fractures

- Develop perpendicular to minimum in-situ stress
- Fractures along the length of the wellbore
- Planar fractures from the perforation
Coalescence of Longitudinal Fractures

- Propagation and coalescence from a horizontal well

Radius of initial fracture = 0.5 m

\( h = 2 \text{ m} \)
\( p = 3.5 \text{ MPa} \)
Coalescence of 3-D Fractures: GFEM Model

- Input mesh and fracture surfaces for GFEM simulation
- Automatic adaptive mesh refinement performed at each propagation step
Coalescence of 3-D Fractures
Coalescence of 3-D Fractures

Fractures just prior to coalescence

Fractures just after coalescence
Coalescence of 3-D Fractures

- Adaptive refinement along fracture fronts
Ongoing Work

- Coalescence of non-planar fractures near a wellbore

Vertical well with initial fractures in a spiral pattern

After Propagation and coalescence

Non-Planar coalesced surface

Initial fracture reorientation

Sharp kink
Crack Coalescence – Non-Planar Cracks

Traction BC on top and bottom surface

PropagationMovie1
Conclusions and Outlook

- Generalized FEM removes several limitations of std FEM
- It enables the solution of problems that are difficult or not practical with the FEM
- This is the case of three-dimensional fracture problems involving
  - Complex crack surfaces
  - Fluid-induced fracturing
  - Coalescence of 3-D fractures, etc.
- Ongoing
  - Coalescence of non-planar fractures
  - Coupling porous medium deformation with fluid flow on fracture
  - Interaction between hydraulic and natural fractures
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