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Improved Conditioning and Accuracy of GFEM/XFEM for Three-Dimensional Fracture Mechanics

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Computational Fracture Mechanics Track
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Acknowledgements

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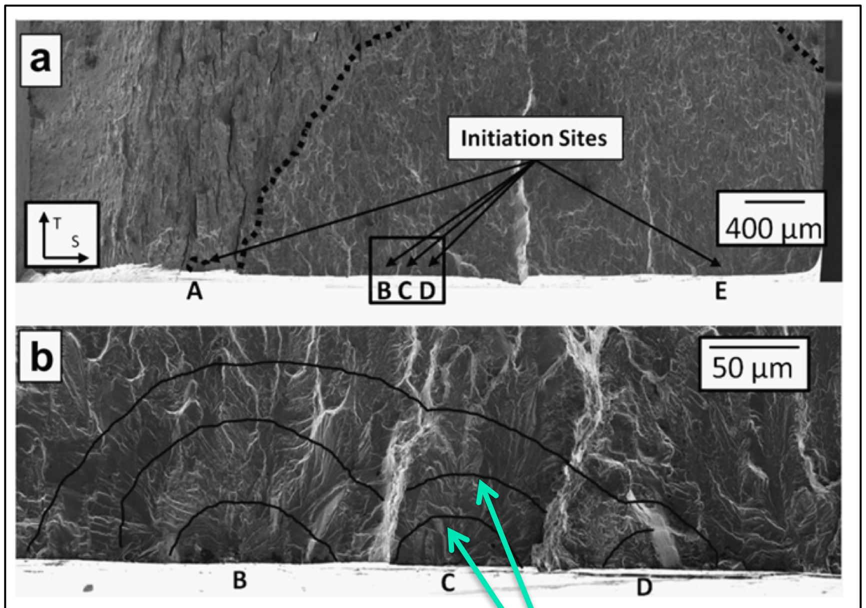
Motivation: Simulation of 3-D Crack Coalescence

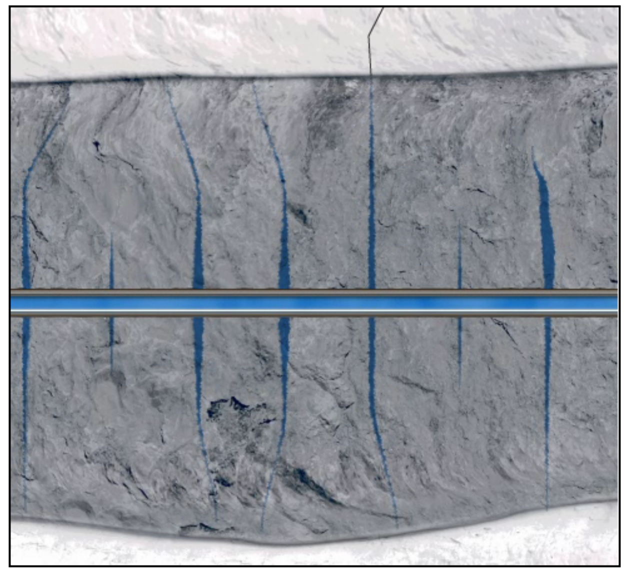
- Modeling and simulation of crack coalescence is of great importance in many applications

Coalescence of fatigue micro-cracks



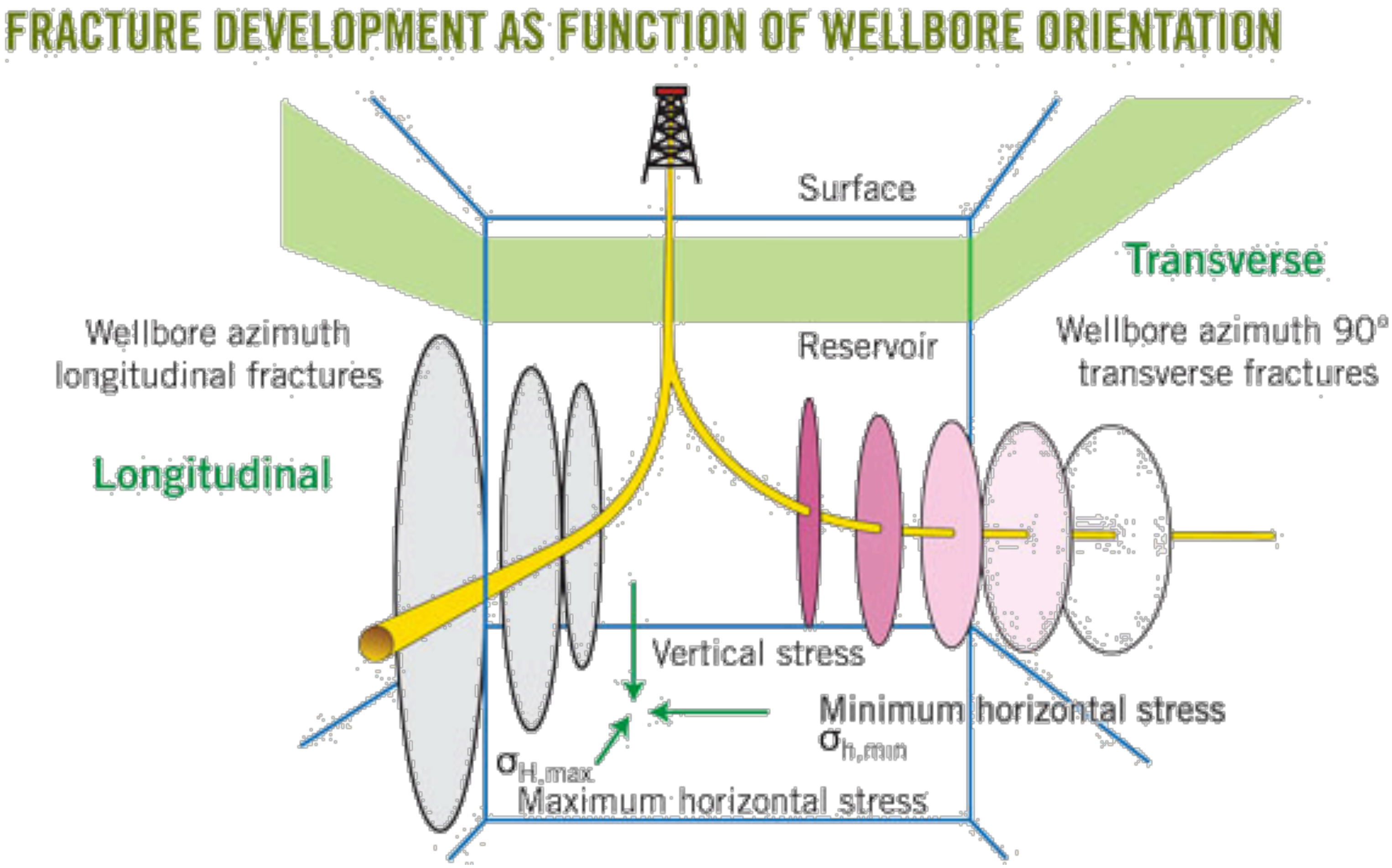
Crack fronts

Hydraulic fractures from horizontal well



Reflective crack in asphalt overlay

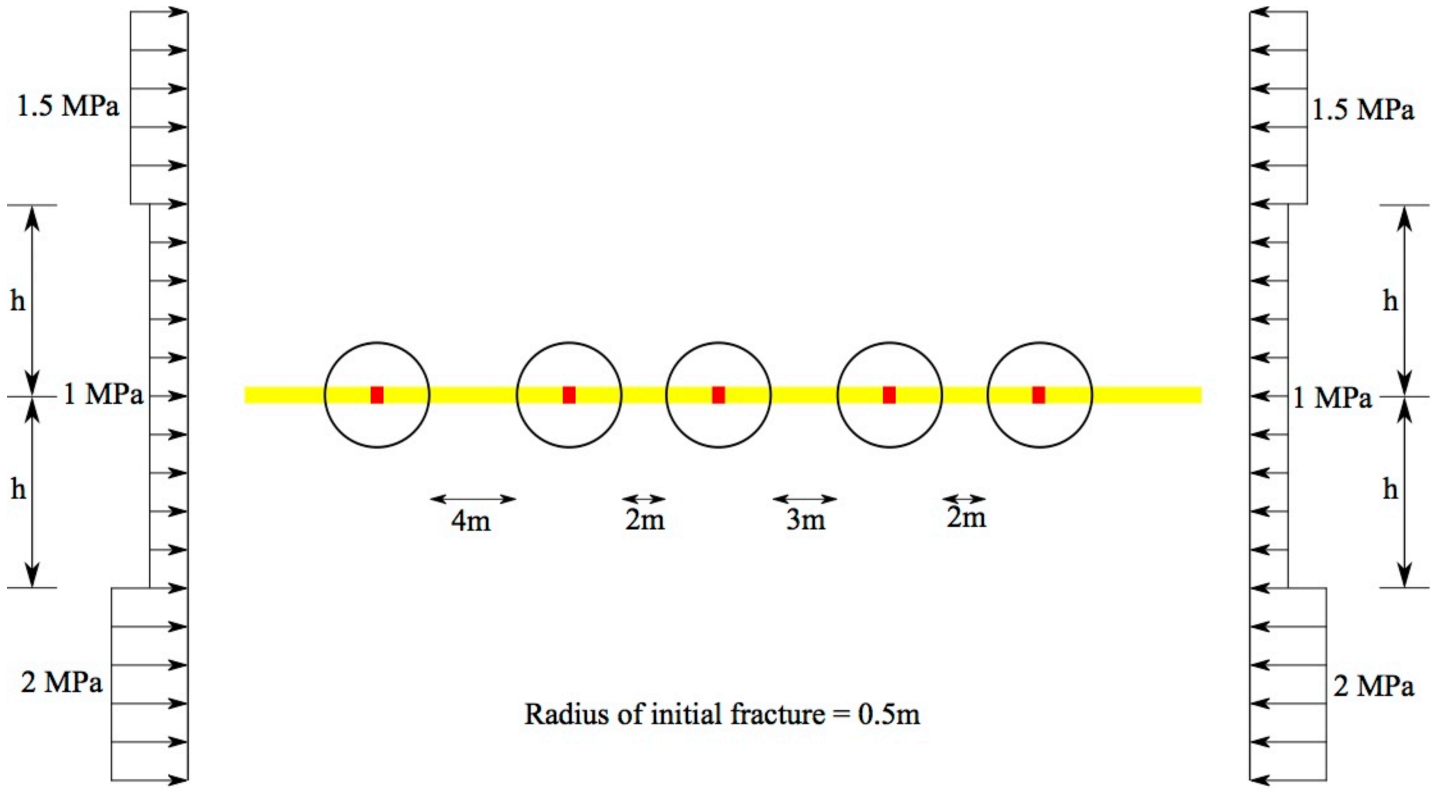
Coalescence of Hydraulic Fractures



[Z. Rahim et al., 2012]

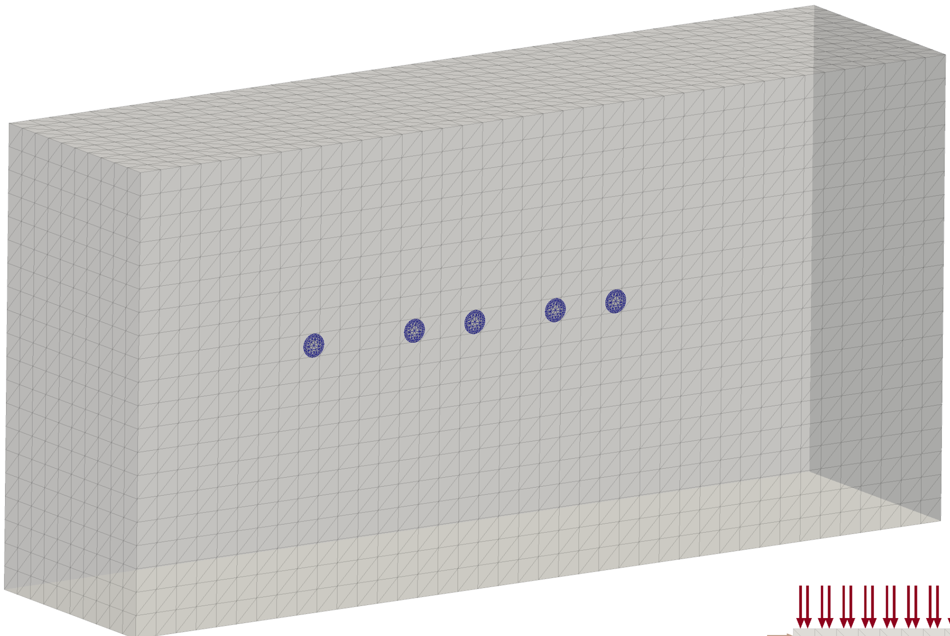
Coalescence of Longitudinal Fractures

- Propagation and coalescence from a horizontal well

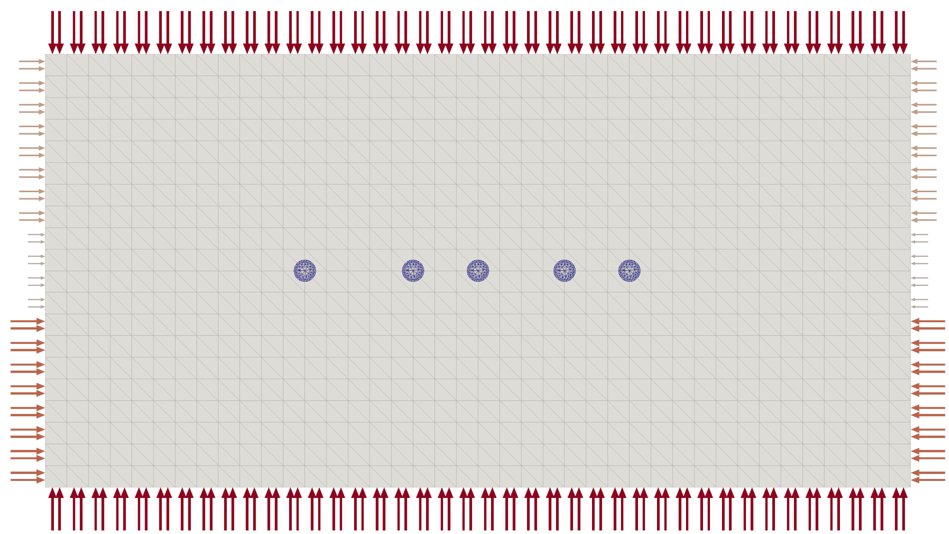


$h = 2$ m
$p = 3.5$ MPa

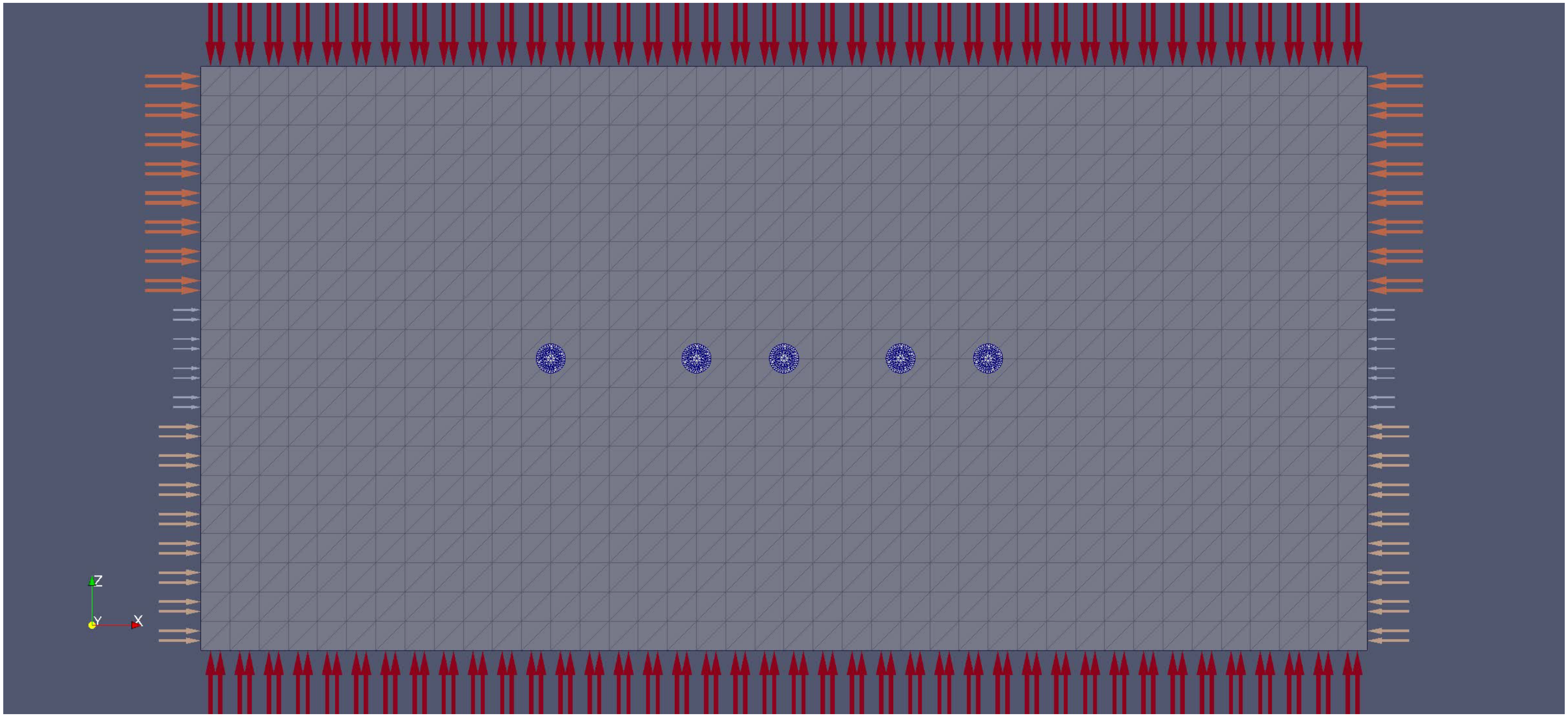
Coalescence of 3-D Fractures: GFEM Model



- Input mesh and fracture surfaces for GFEM simulation

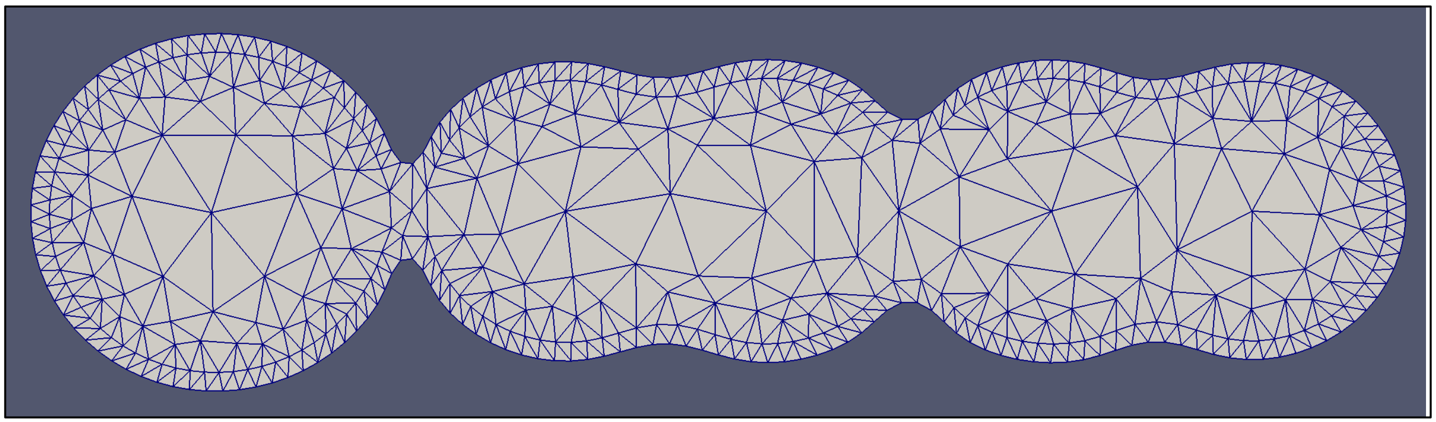


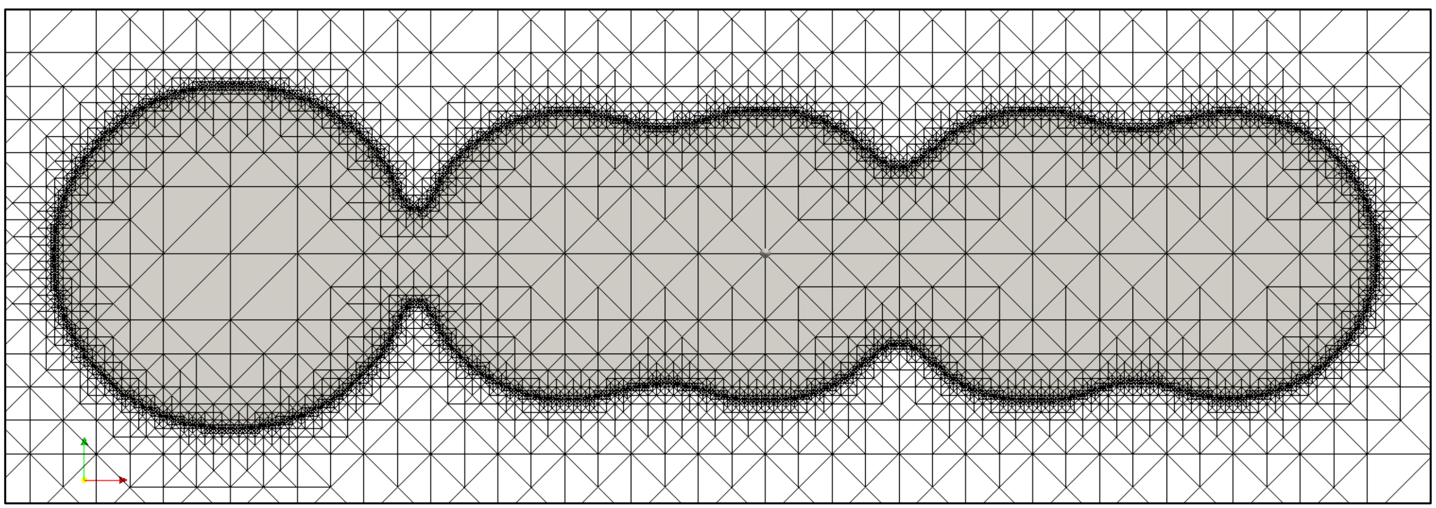
Coalescence of 3-D Fractures



Coalescence of 3-D Fractures

- A large 3-D model is required even with adaptive refinement along fracture fronts
- Numerical conditioning of the G/XFEM becomes critical





Outline

- Motivation
- Basic ideas of G/XFEM
- Stable GFEM for 3D fractures
- Assessment of convergence and numerical conditioning
- Conclusions and future work

Early Works on Generalized FEMs

- Babuska, Caloz and Osborn, 1994 (Special FEM).
- Duarte and Oden, 1995 (Hp Clouds).
- Babuska and Melenk, 1995 (PUFEM).
- Oden, Duarte and Zienkiewicz, 1996 (Hp Clouds/GFEM).
- Duarte, Babuska and Oden, 1998 (GFEM).
- Belytschko et al., 1999 (Extended FEM).
- Strouboulis, Babuska and Copps, 2000 (GFEM).

- Basic idea:
  - Use a partition of unity to build Finite Element shape functions

- Review paper

  Belytschko T., Gracie R. and Ventura G. A review of extended/generalized finite element methods for material modeling, *Mod. Simul. Matl. Sci. Eng.,* 2009

  "The XFEM and GFEM are basically identical methods: the name generalized finite element method was adopted by the Texas school in 1995–1996 and the name extended finite element method was coined by the Northwestern school in 1999."

Generalized Finite Element Method

- GFEM is a Galerkin method with special test/trial space given by

$$\mathbb{S}_{GFEM} = \mathbb{S}_{FEM} + \mathbb{S}_{ENR}$$

Low order FEM space

Enrichment space with functions related to the given problem

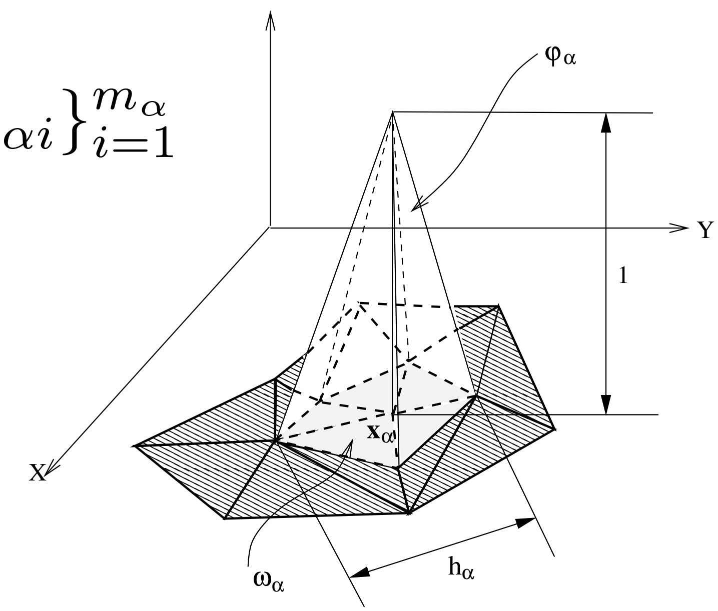
$$\mathbb{S}_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R}$$

$$\mathbb{S}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \mathrm{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}$$

$$L_{\alpha i} \in \chi_\alpha(\omega_\alpha)$$

Enrichment function

Patch space



Generalized Finite Element Method

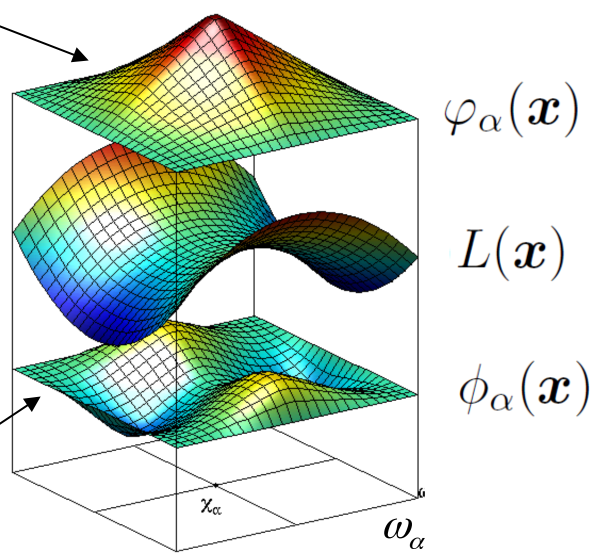
$$\mathbb{S}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \chi_\alpha; \quad \chi_\alpha = \mathrm{span}\{L_{\alpha i}\}_{i=1}^{m_\alpha}$$

$$\phi_{\alpha i}(x) = \varphi_\alpha(x) L_{\alpha i}(x) \qquad \sum_\alpha \varphi_\alpha(\boldsymbol{x}) = 1$$

Linear FE shape function

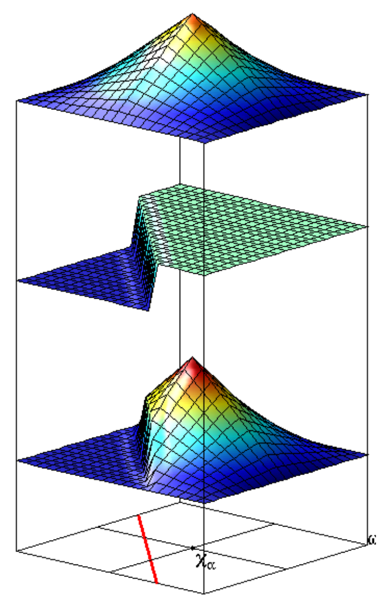
Enrichment function

GFEM shape function



[Oden, Duarte & Zienkiewicz, 1996]

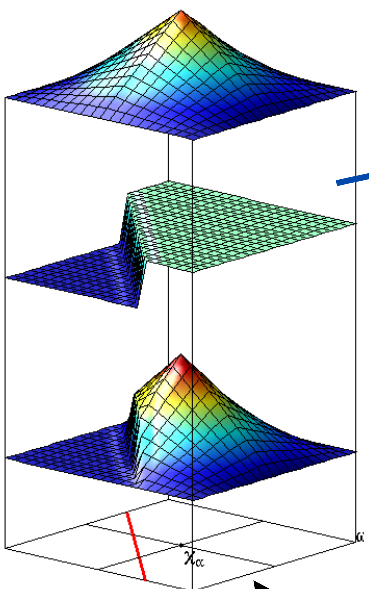
- Allows construction of shape functions incorporating a-priori knowledge about solution



Discontinuous enrichment [Moes et al., 1999]

GFEM Approximation for 3-D Fractures

$$\mathbb{S}_{GFEM}(\Omega) = \left\{ \boldsymbol{u}^{hp} = \sum_{\alpha \in I_h} \underbrace{\varphi_\alpha(\boldsymbol{x})}_{\text{PoU}} \left[\underbrace{\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{polynomial}} + \underbrace{\mathcal{H}\tilde{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{discontinuous}} + \underbrace{\breve{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{singular}} \right] \right\}$$

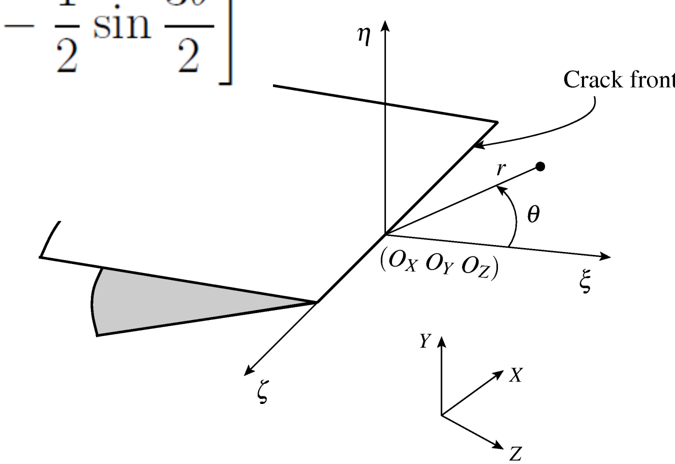


$$\breve{L}^{\xi}_{\alpha 1}(r,\theta) = \sqrt{r}\left[(\kappa - \frac{1}{2})\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right]$$

$$\breve{L}^{\eta}_{\alpha 1}(r,\theta) = \sqrt{r}\left[(\kappa + \frac{1}{2})\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2}\right]$$

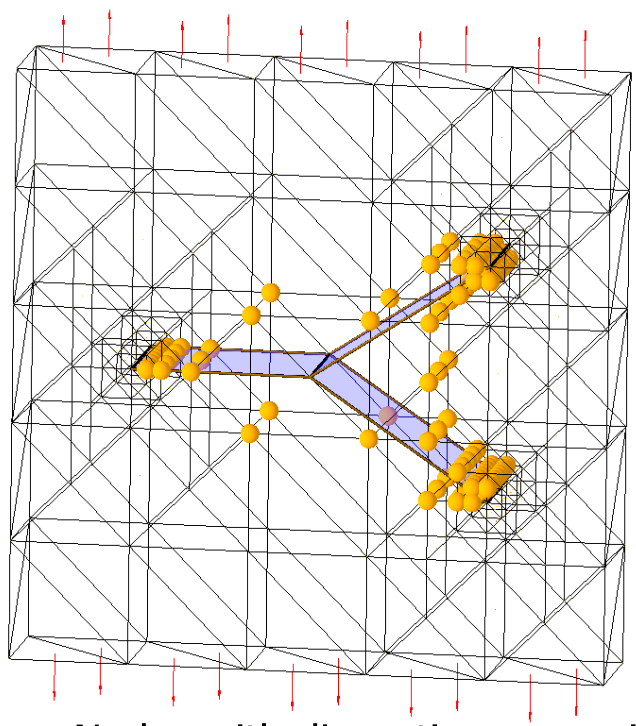
$$\breve{L}^{\zeta}_{\alpha 1}(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}$$

[Duarte & Oden 1996]

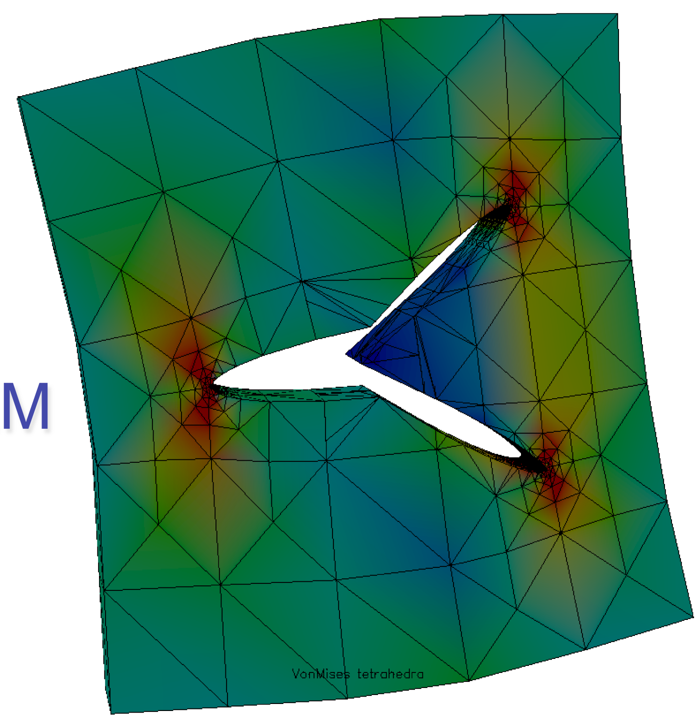


Modeling Fractures with the GFEM

- Discontinuities modeled via enrichment functions, *not* the FEM mesh
- Mesh refinement *still required* for acceptable accuracy



hp-GFEM



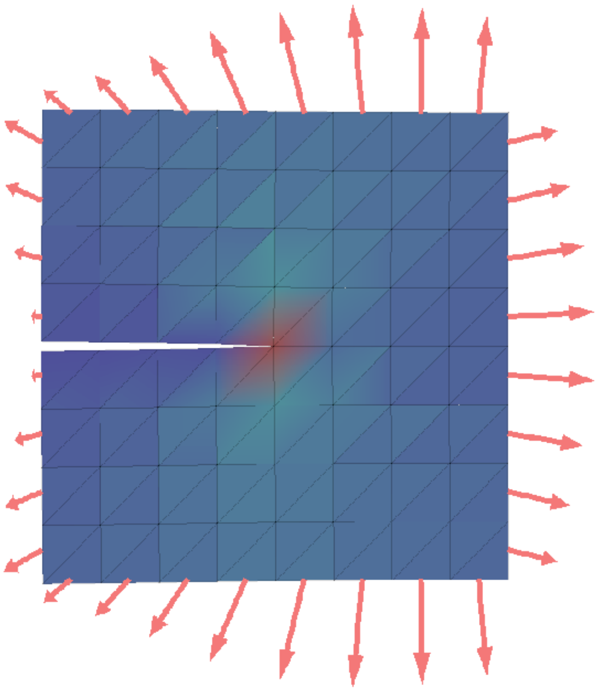
● = Nodes with discontinuous enrichments

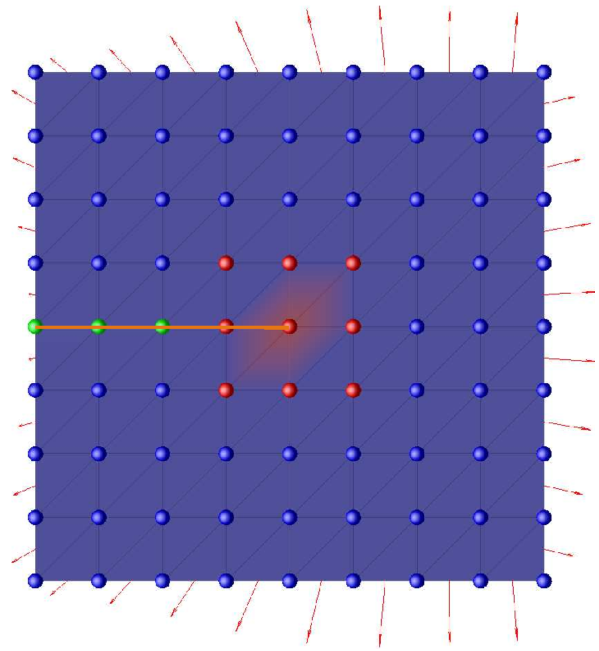
Von Mises stress

[Duarte et al., International Journal Numerical Methods in Engineering, 2007]

Conditioning of G/XFEM Approximations

- 2-D edge-crack panel loaded with Mode I tractions





- Red nodes: Singular enrichments
- Green nodes: Heaviside function

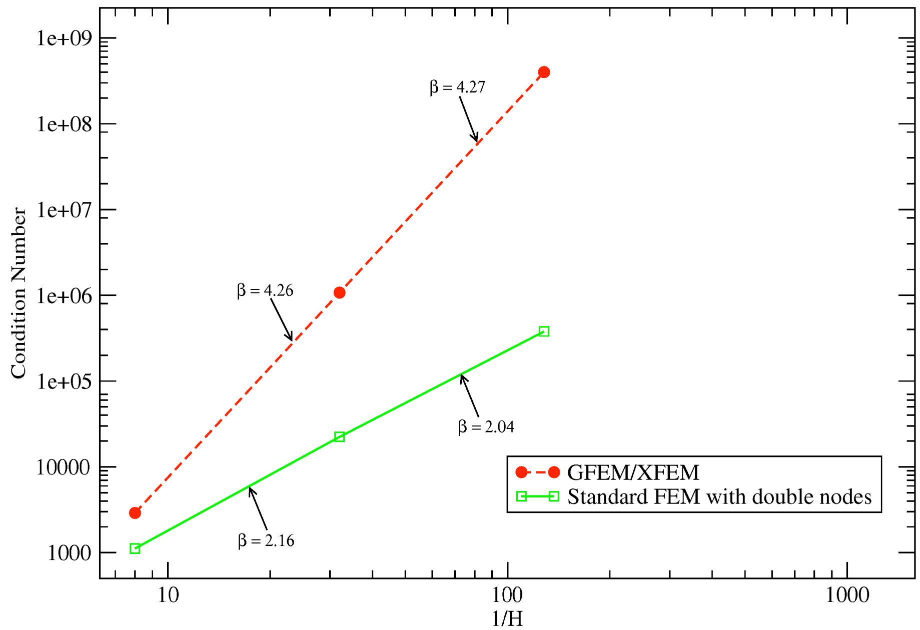
$$\mathbf{u}_I(r,\theta) = \sqrt{r}\begin{Bmatrix} \left(\kappa - \frac{1}{2}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2} \\ \left(\kappa + \frac{1}{2}\right)\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2} \end{Bmatrix}$$

where $\kappa = 3\text{-}4\nu$

ν is the Poisson's ratio

$$\mathscr{H}(x,y) = \begin{cases} 1 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

Conditioning of G/XFEM Approximations



- The conditioning of the G/XFEM stiffness matrix, $\boldsymbol{K}_{GFEM}$, can be much worse than that of the standard FEM, $\boldsymbol{K}_{FEM}$

$$\mathfrak{K}(\boldsymbol{K}_{GFEM}) = \mathcal{O}(h^{-4})$$

while

$$\mathfrak{K}(\boldsymbol{K}_{FEM}) = \mathcal{O}(h^{-2})$$

where $\mathfrak{K}(.)$ is the *scaled condition number*.

Stable GFEM (SGFEM)*

- Goal: Control the conditioning of GFEM/XFEM while preserving approximation properties
- Involves simple modification of enrichment functions
- Performs near-orthogonalization of enrichments w.r.t. Finite Element Partition of Unity
- Straightforward to implement in an existing GFEM/XFEM code
- Bonus: Increased accuracy at no additional cost!

* I. Babuska and U. Banerjee. Stable Generalized Finite Element Method (SGFEM). *CMAME*, 2012.

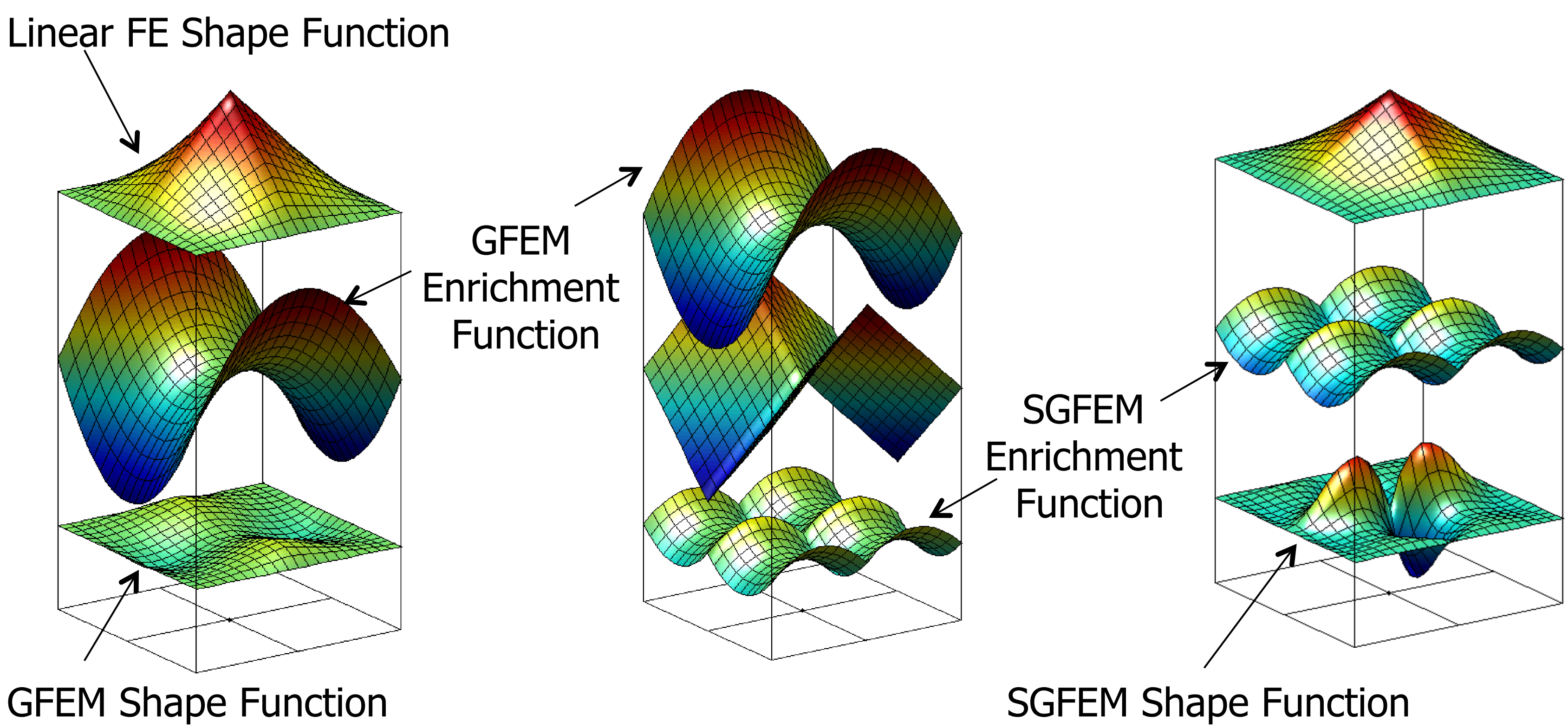
* V. Gupta, C.A. Duarte, I. Babuska and U. Banerjee. A Stable and Optimally Convergent Generalized FEM (SGFEM) for Linear Elastic Fracture Mechanics. *CMAME*, 2013, 2014 (submitted).

SGFEM: Stable Generalized FEM

Modification of enrichment functions

$$\tilde{L}_{\alpha i}(\boldsymbol{x}) = L_{\alpha i}(\boldsymbol{x}) - \mathrm{I}_{\omega_\alpha}(L_{\alpha i})(\boldsymbol{x})$$

$$\tilde{\phi}_{\alpha i}(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})\tilde{L}_{\alpha i}(\boldsymbol{x})$$



Stable Generalized Finite Element Method

$$\mathbb{S}_{SGFEM} = \mathbb{S}_{FEM} + \widetilde{\mathbb{S}}_{ENR}$$

$$\mathbb{S}_{FEM} = \sum_{\alpha \in I_h} c_\alpha \varphi_\alpha, \quad c_\alpha \in \mathbb{R}$$

$$\widetilde{\mathbb{S}}_{ENR} = \sum_{\alpha \in I_h^e \subset I_h} \varphi_\alpha \widetilde{\chi}_\alpha; \quad \widetilde{\chi}_\alpha = \mathrm{span}\{\tilde{L}_{\alpha i}\}_{i=1}^{m_\alpha}$$

$\tilde{L}_{\alpha i} \in \widetilde{\chi}_\alpha(\omega_\alpha)$ Modified enrichment functions

$$\tilde{L}_{\alpha i}(\boldsymbol{x}) = L_{\alpha i}(\boldsymbol{x}) - \mathrm{I}_{\omega_\alpha}(L_{\alpha i})(\boldsymbol{x})$$

Stable Generalized Finite Element Method

The conditioning of the SGFEM matrix is of same order as that of standard FEM

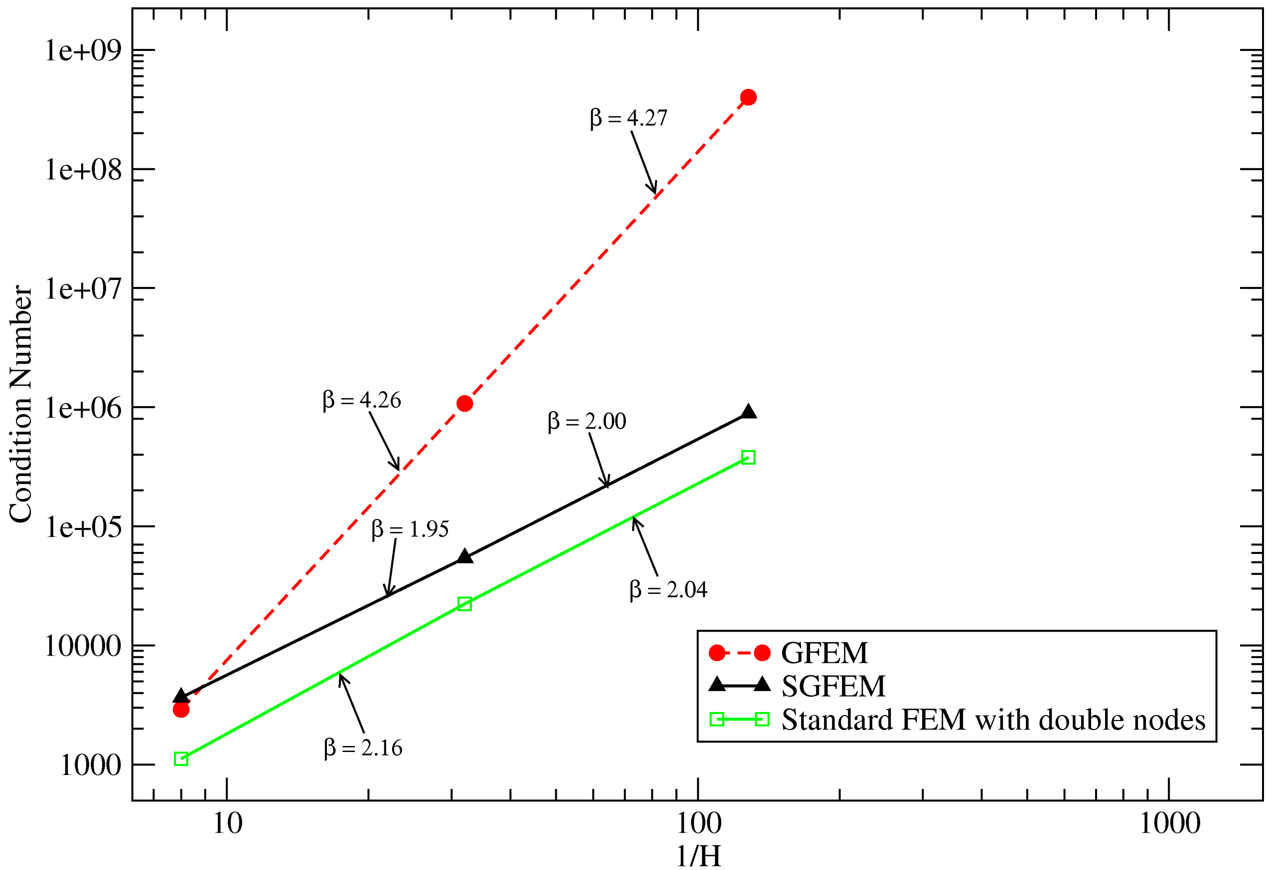
$$\mathfrak{K}(\boldsymbol{K}_{\mathrm{SGFEM}}) = \mathfrak{K}(\boldsymbol{K}_{\mathrm{FEM}}) = \mathcal{O}(h^{-2})$$

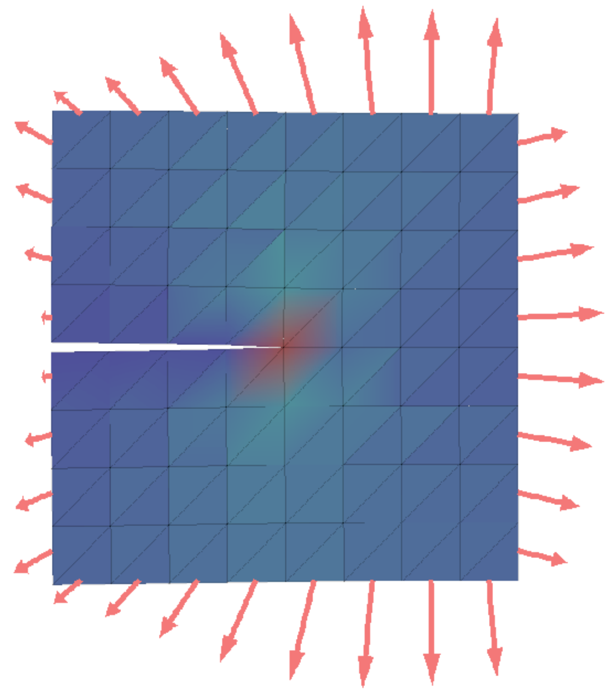
$$\mathbb{S}_{SGFEM} = \mathbb{S}_{FEM} + \widetilde{\mathbb{S}}_{ENR}$$

$$\boldsymbol{K}_{\mathrm{SGFEM}} = \begin{bmatrix} \boldsymbol{K}_{\mathrm{FEM}} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{\mathrm{ENR}} \end{bmatrix}$$

Property 1: the spaces $\widetilde{\mathbb{S}}_{ENR}$ and $\mathbb{S}_{FEM}$ are *almost orthogonal* with respect to the energy inner product $B(\cdot,\cdot)$;
Property 2: the eigenvalues of the diagonally scaled matrix of $\boldsymbol{K}_{\mathrm{ENR}}$ are bounded away from 0.

SGFEM: Stable Generalized FEM





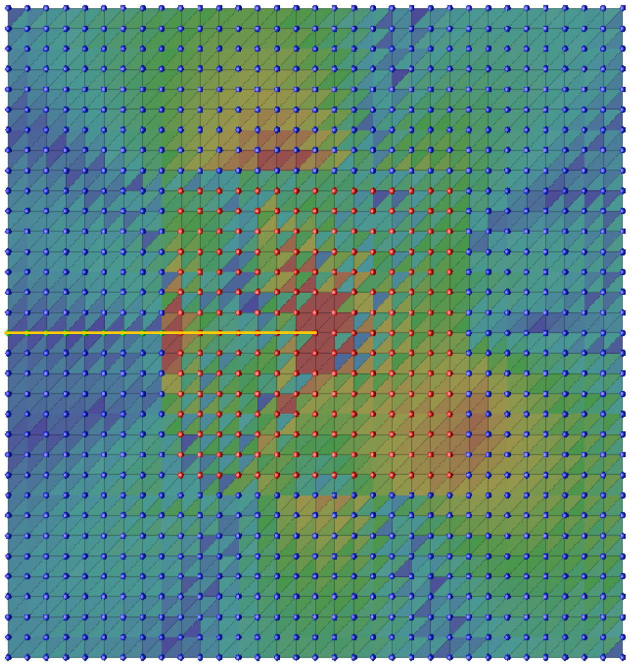
Conditioning of GFEM/XFEM stiffness matrix $\mathcal{O}(h^{-4})$

Conditioning of SGFEM and FEM stiffness matrix $\mathcal{O}(h^{-2})$

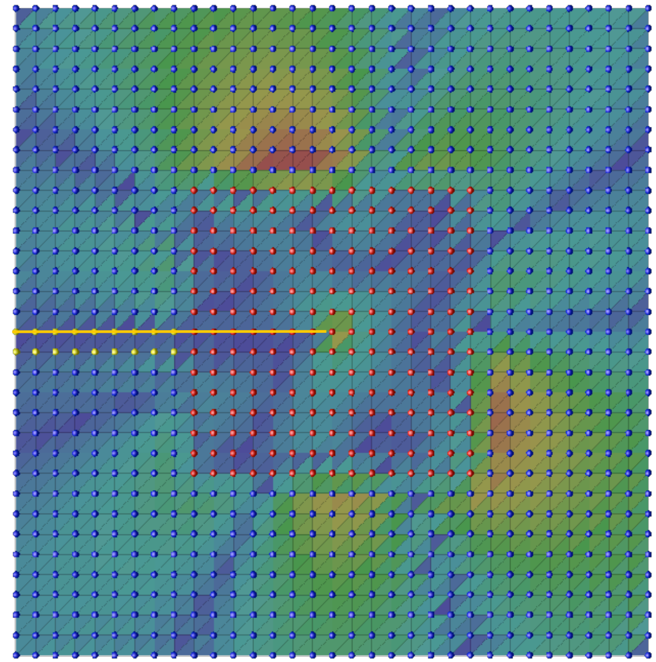
[Gupta, Duarte, Babuska & Banerjee CMAME, 2013]

GFEM/XFEM vs SGFEM: Accuracy

Element-wise error in energy norm



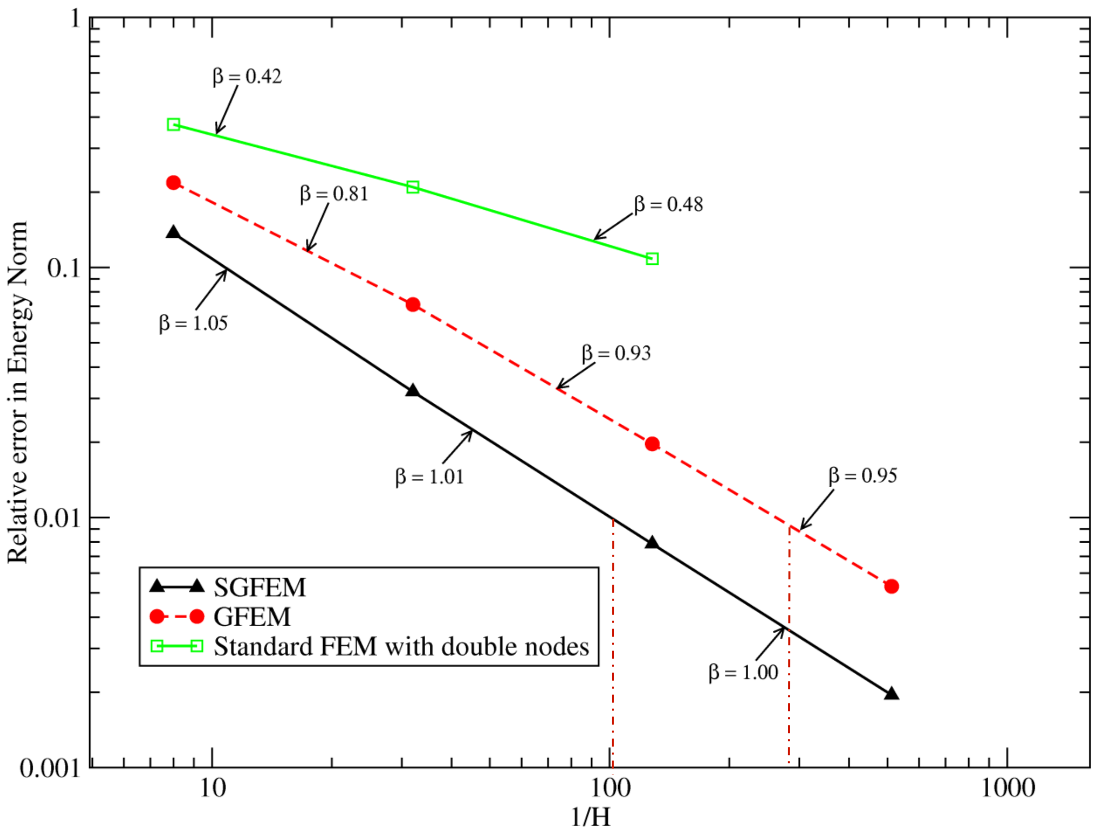
GFEM/XFEM
(Green nodes: Heaviside Enrichment)



SGFEM
(Yellow nodes: Linear Heaviside Enrichment)

- SGFEM shows lower error in the entire enrichment zone
- GFEM and SGFEM: Optimal O(h) convergence

GFEM/XFEM vs SGFEM: Global Convergence



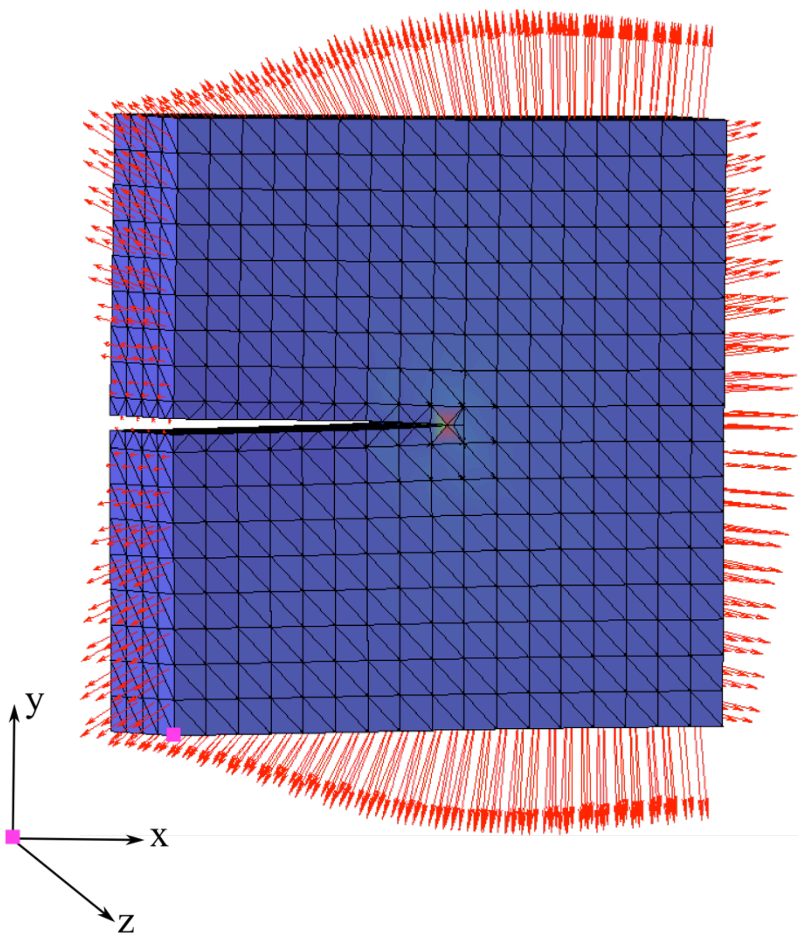
Mesh size for ~1% Error :

- GFEM\XFEM: H = 0.003
- SGFEM: H = 0.01

- GFEM and SGFEM: Optimal O(h) convergence
- FEM: $O(h^{1/2})$ convergence

SGFEM for 3-D Fracture

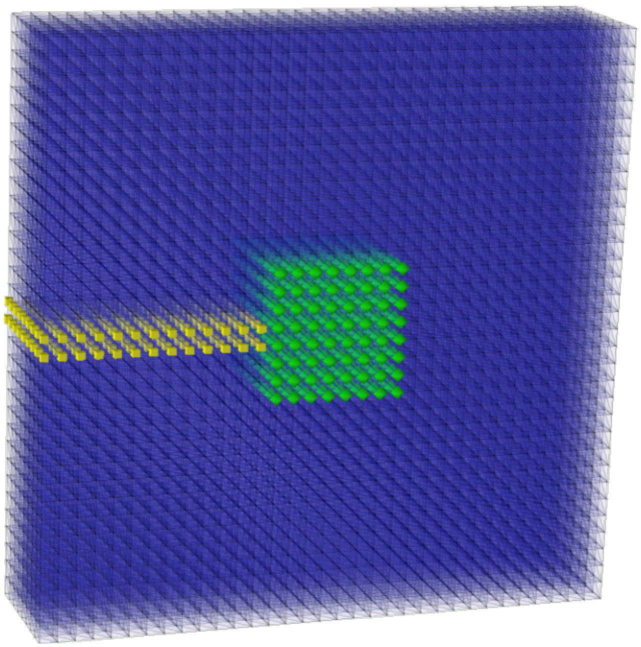
- Quasi 3-D edge-crack panel loaded with Mode I tractions



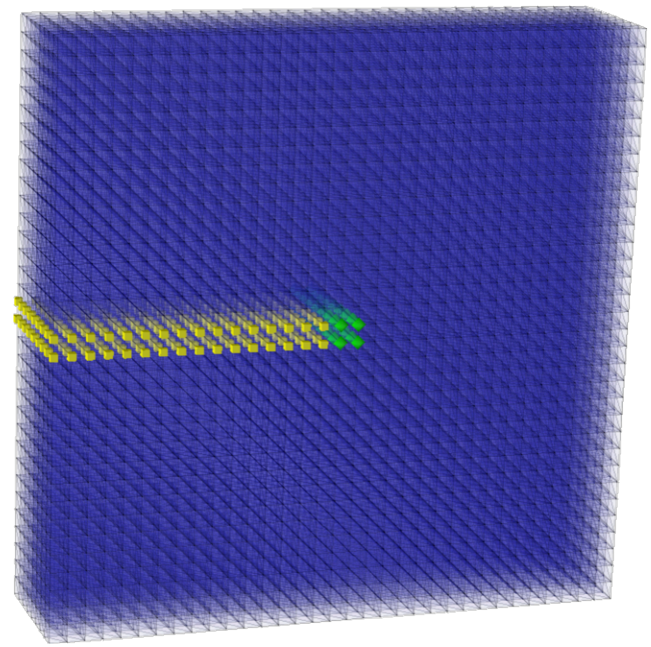
$$\mathbf{u}_I(r,\theta) = \sqrt{r}\begin{Bmatrix} \left(\kappa - \frac{1}{2}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2} \\ \left(\kappa + \frac{1}{2}\right)\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2} \\ 0 \end{Bmatrix}$$

➢ Solution is constant in z-direction

Geometrical and Topological Enrichments



- Green nodes: Singular fn
- Yellow nodes: Heaviside fn



Geometrical Enrichment

- Enrich all nodes within a fixed sub-domain around crack front
- May lead to large number of dofs if global mesh is fine
- Leads to ill-conditioned system

Topological Enrichment

- Only elements cut/touched by crack front are enriched
- Enrichment domain depends on mesh density
- Leads to sub-optimal convergence

Singular Enrichments are not Unique

OD Singular basis: Oden and Duarte, 2000

$$\mathbf{L}^{OD}_{front-\bar{x}} = \left\{\sqrt{r}\left[\left(\kappa-\frac{1}{2}\right)\cos\frac{\theta}{2}-\frac{1}{2}\cos\frac{3\theta}{2}\right], \sqrt{r}\left[\left(\kappa+\frac{3}{2}\right)\sin\frac{\theta}{2}+\frac{1}{2}\sin\frac{3\theta}{2}\right]\right\}$$

$$\mathbf{L}^{OD}_{front-\bar{y}} = \left\{\sqrt{r}\left[\left(\kappa+\frac{1}{2}\right)\sin\frac{\theta}{2}-\frac{1}{2}\sin\frac{3\theta}{2}\right], \sqrt{r}\left[\left(\kappa-\frac{3}{2}\right)\cos\frac{\theta}{2}+\frac{1}{2}\cos\frac{3\theta}{2}\right]\right\}$$

$$\mathbf{L}^{OD}_{front-\bar{z}} = \left\{\sqrt{r}\left[\sin\frac{\theta}{2}\right], r^2\left[\cos 2\theta\right]\right\}$$

- 6 enrichments per node
- Referred to as *vector* enrichments

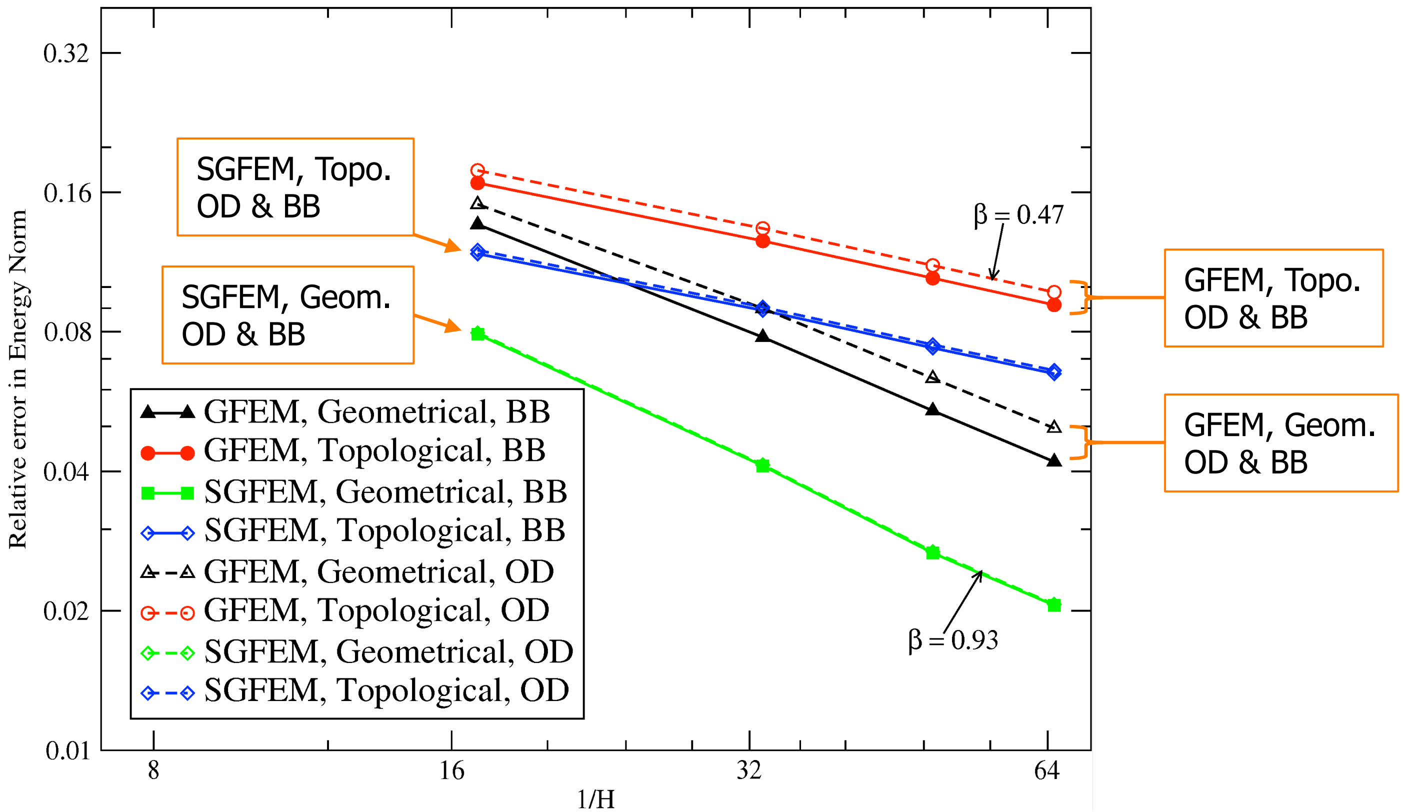
BB Singular basis: Belytschko and Black, 1999

$$\mathbf{L}^{BB}_{front} = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\} \quad \text{in } x, y, z$$

- 12 enrichments per node
- Referred to as *scalar* enrichments

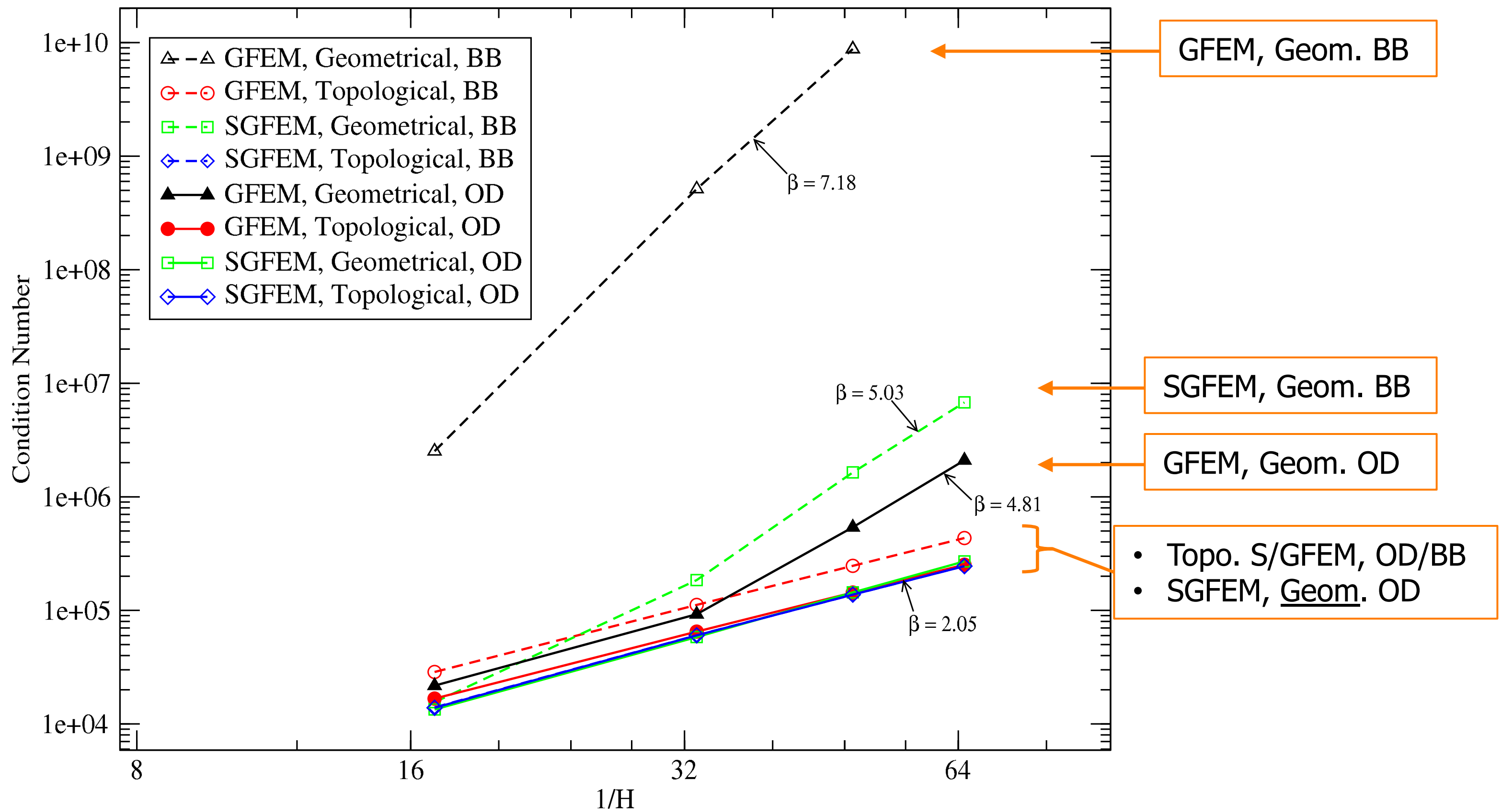
➢ Both bases span a space containing the exact solution

GFEM/XFEM vs SGFEM: Convergence of OD or BB



- SGFEM yields better accuracy than GFEM even for topological enrichment
- Error of GFEM with OD > Error of GFEM with BB
- Error of SGFEM with OD ~= Error of SGFEM with BB

GFEM/XFEM vs SGFEM: Conditioning (OD or BB)



- BB (scalar) enrichments yield much higher growth in conditioning
- SGFEM improves conditioning for both OD and BB enrichments
- SGFEM with OD basis yields similar conditioning as standard FEM: ~ $O(h^{-2})$

Fully 3-D Mode-I Expansion (OY)*

$$\tilde{\mathbf{u}}_I = \begin{Bmatrix} u_r \\ u_\theta \\ u_{\bar{z}} \end{Bmatrix} = A_1 r^{\frac{1}{2}} \begin{Bmatrix} (Q_1 - 1)\sin\frac{\check{\theta}}{2} + \sin\frac{3\check{\theta}}{2} \\ -(Q_1+1)\cos\frac{\check{\theta}}{2} - \cos\frac{3\check{\theta}}{2} \\ 0 \end{Bmatrix} + \frac{dA_1}{d\check{z}} r^{\frac{3}{2}} \begin{Bmatrix} 0 \\ 0 \\ 2\sin\frac{\check{\theta}}{2} + \frac{2}{3}(Q_1+1)\sin\frac{3\check{\theta}}{2} \end{Bmatrix}$$

$$+ \frac{d^2 A_1}{d\check{z}^2} r^{\frac{5}{2}} \begin{Bmatrix} Q_2 \sin\frac{\check{\theta}}{2} + Q_3 \sin\frac{3\check{\theta}}{2} \\ \frac{1}{6}(Q_1+1)\cos\frac{\check{\theta}}{2} - Q_4\cos\frac{3\check{\theta}}{2} \\ 0 \end{Bmatrix}$$

where

$$\check{z} = -\bar{z}, \quad \check{\theta} = \pi - \theta, \quad Q_1 = \frac{(2\lambda + 6\mu)}{(\lambda+\mu)}, \quad Q_2 = \frac{(3\lambda - \mu)}{6(\lambda+\mu)},$$

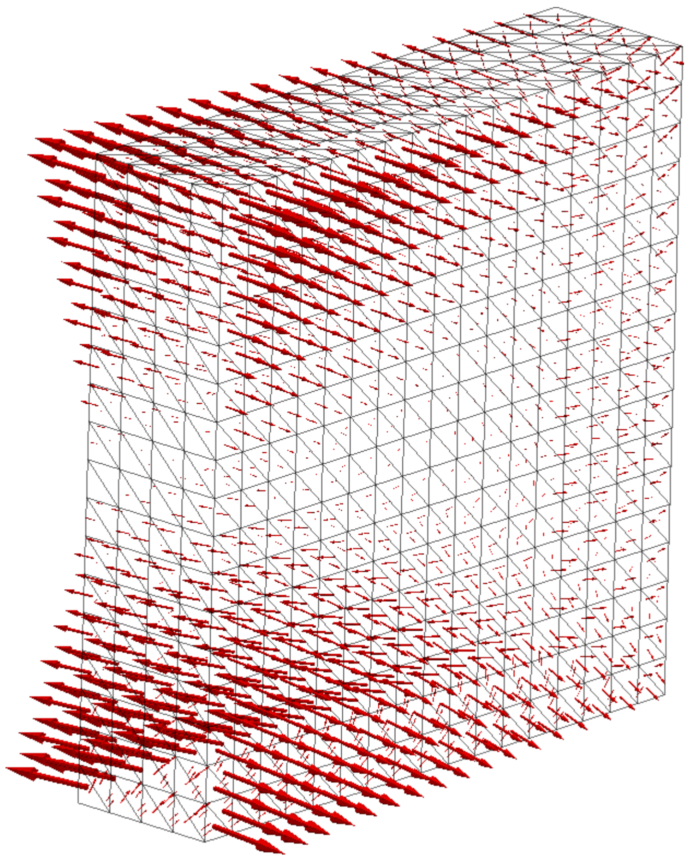
$$Q_3 = \frac{(45\lambda^2 + 138\lambda\mu + 61\mu^2)}{90(\lambda+\mu)^2}, \quad Q_4 = \frac{(-15\lambda^2 + 2\lambda\mu + 49\mu^2)}{90(\lambda+\mu)^2}.$$

- SIF varies quadratically as

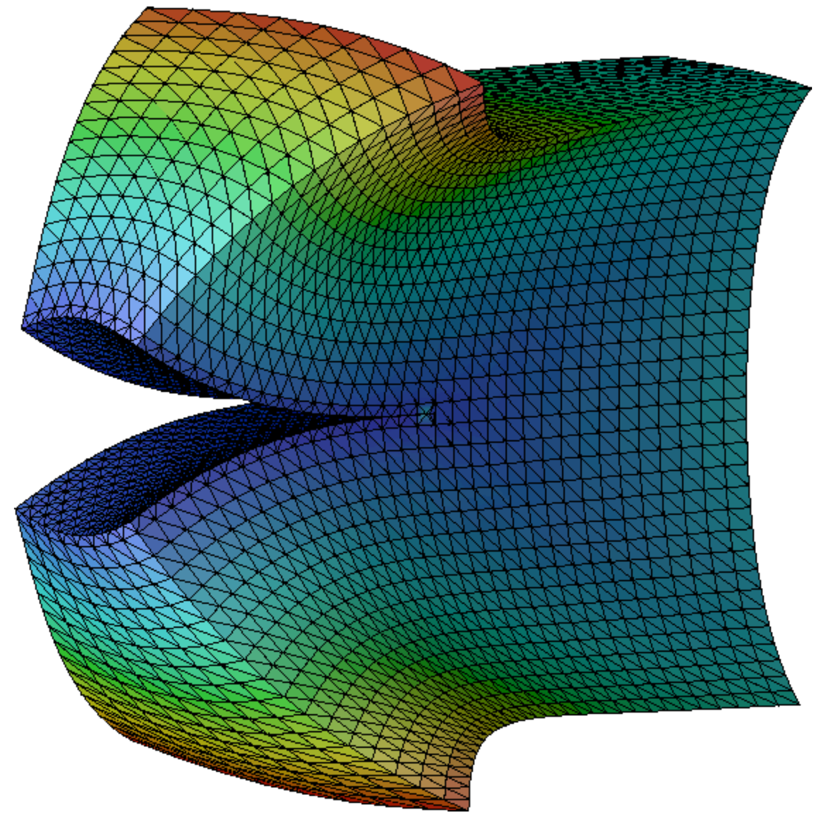
$$K_I = A_1\sqrt{2\pi}\,\frac{2E}{1+\nu}; \quad A_1 = (1-\zeta)*(1+\zeta); \quad \zeta = \frac{\bar{z}}{t_z/2}$$

*[Omer and Yosibash, 2005]

Fully 3-D Edge-Crack Problem

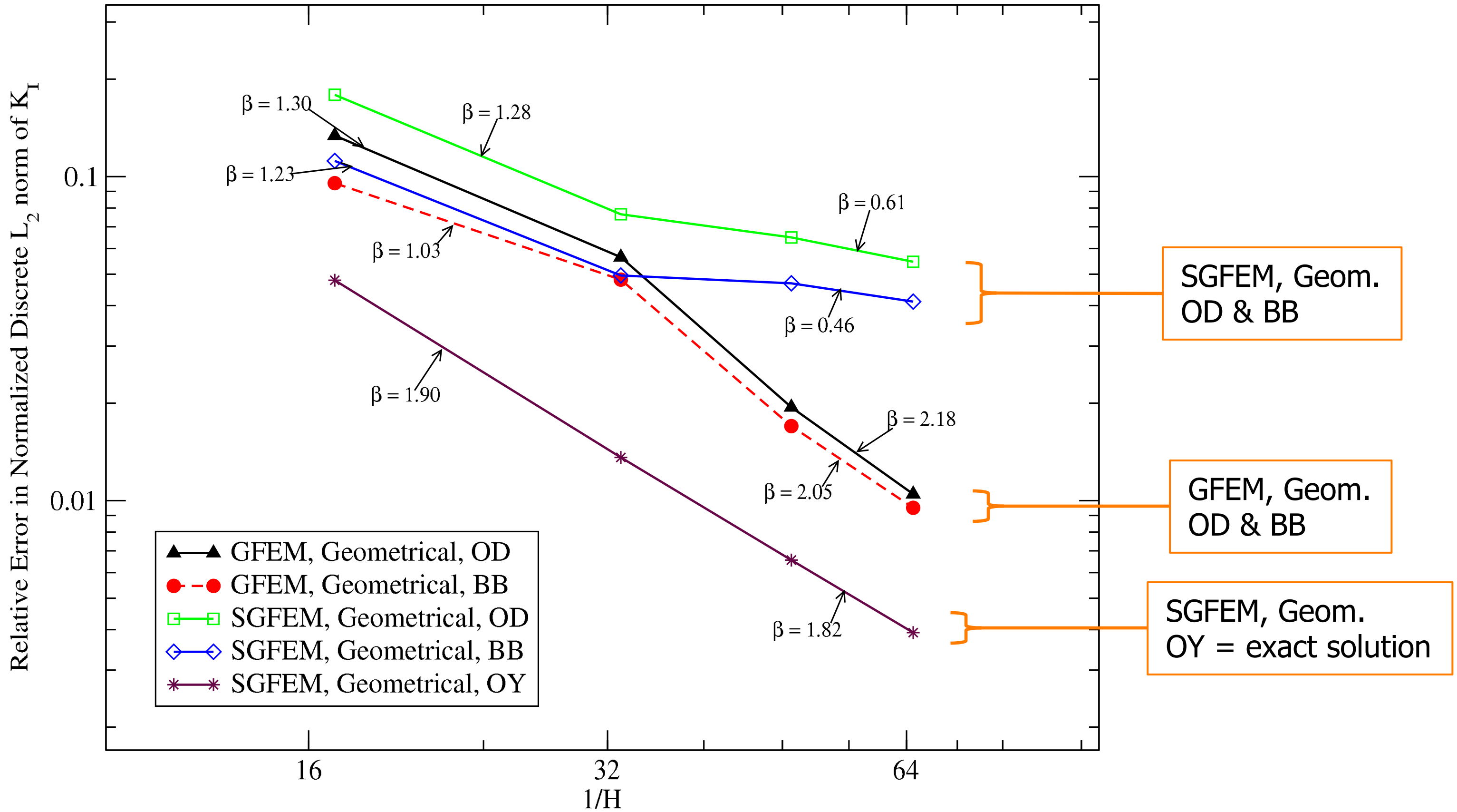


Fully 3-D Mode-I Tractions



Deformed configuration
(von Mises stress distribution)

Convergence of Mode I SIF



$$e^r(K_I) := \frac{\|e_I\|_{L^2}}{\|K_I\|_{L^2}} = \frac{\sqrt{\sum_{j=1}^{N_{ext}} \left(\hat{K}_I^j - K_I^j\right)^2}}{\sqrt{\sum_{j=1}^{N_{ext}} \left(K_I^j\right)^2}}$$

- SGFEM with exact solution (OY) as enrichment: Reference
- Straightforward extension of 2-D singular bases leads to sub-optimal convergence of SGFEM in 3-D

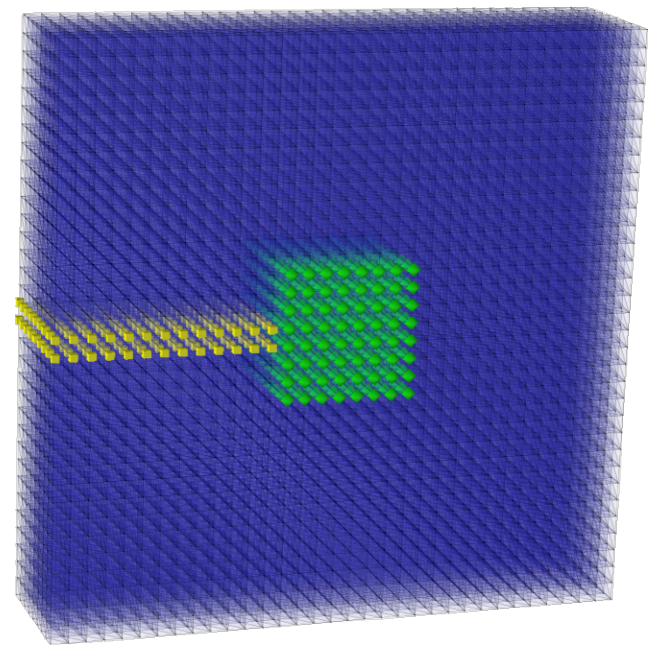
Recovery of Optimal Convergence of SGFEM

- Exact solution is not constant in z-direction but enrichments (BB & OD) are.
- Exact solution is smooth in z-direction (SIFs are smooth functions of z).
- Add linear enrichments on nodes with singular basis to recover optimal convergence with SGFEM

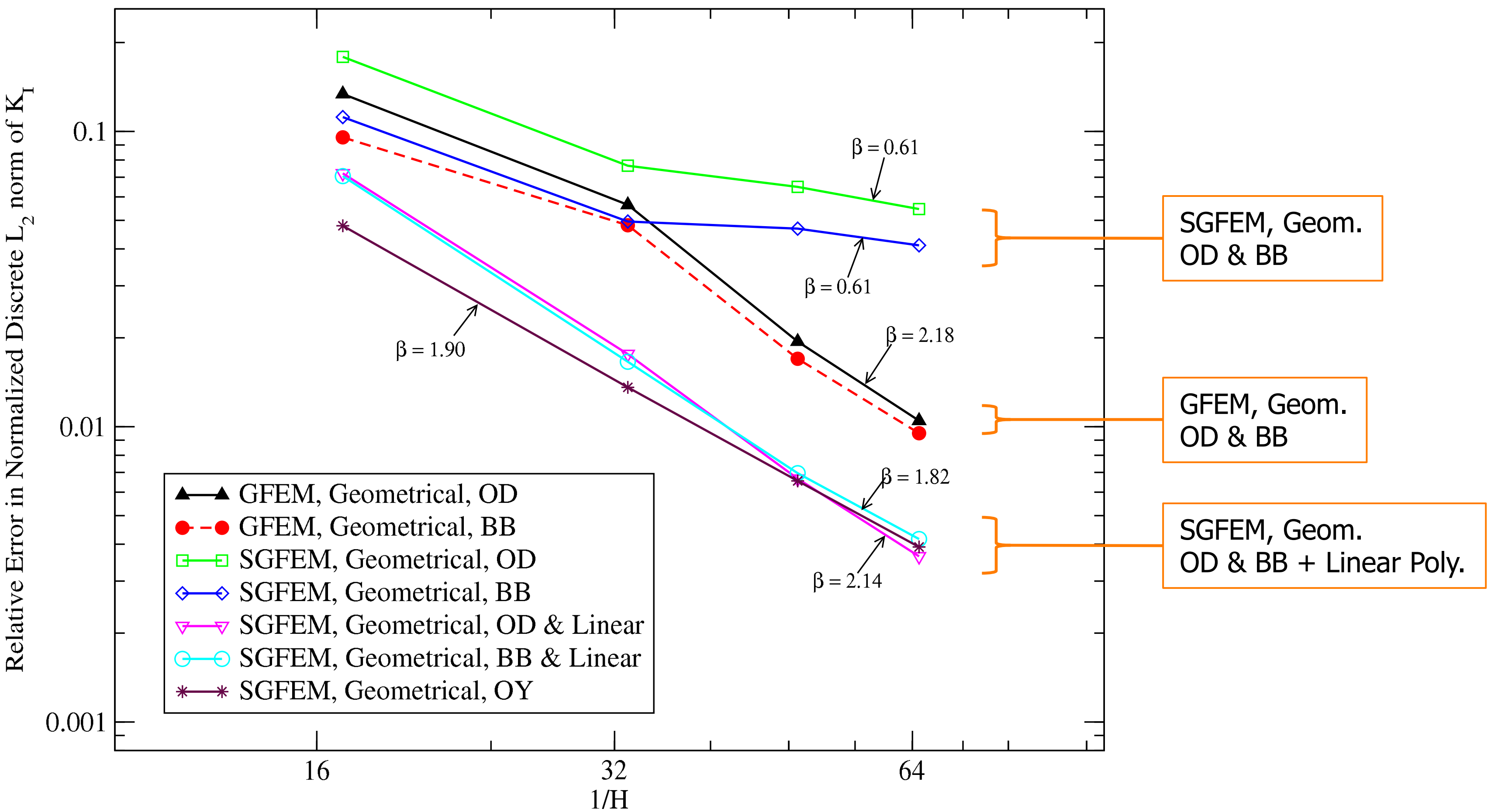
$$\boldsymbol{L}_{\alpha}^{\mathrm{lin}} = \left\{ \frac{(x-x_{\alpha})}{h_{\alpha}}, \frac{(y-y_{\alpha})}{h_{\alpha}}, \frac{(z-z_{\alpha})}{h_{\alpha}} \right\}$$

❑ Green nodes: Singular fns + linear polynomials

❑ Yellow nodes: Heaviside fn

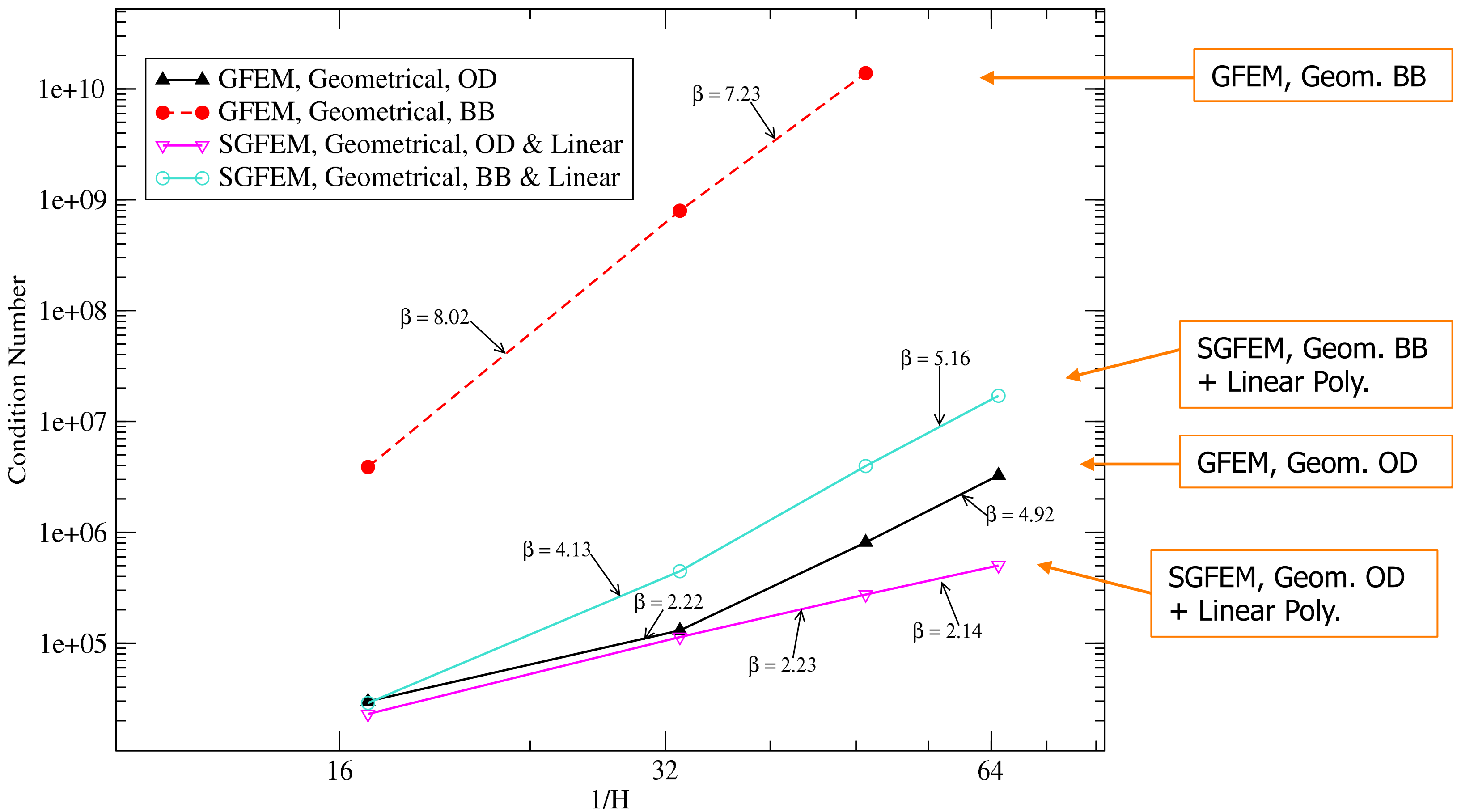


Recovery of Optimal Convergence of SGFEM



- Adding linear enrichments on nodes with singular basis recover optimal convergence of SGFEM
- What about conditioning?

SGFEM for 3-D Fracture: Conditioning



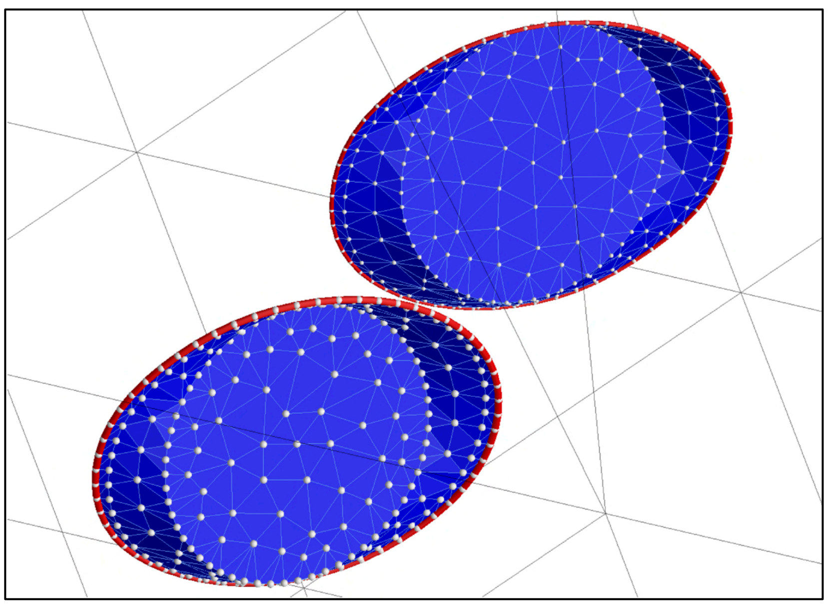
- Adding linear enrichments *does not impact growth of conditioning of SGFEM*

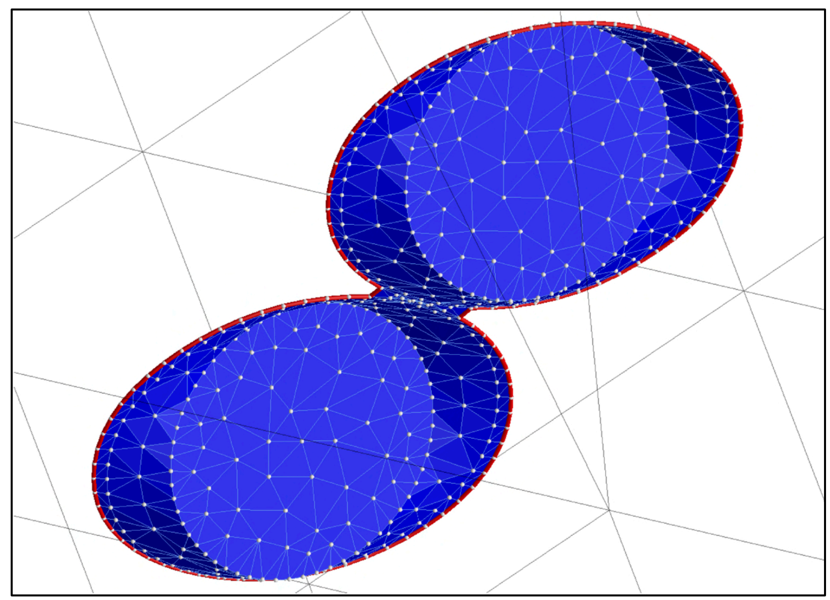
Conclusions

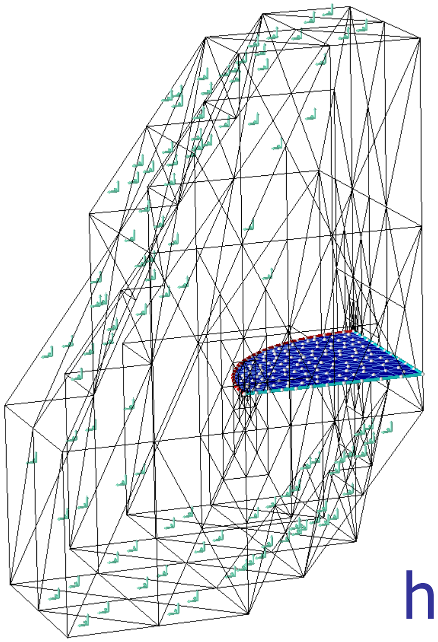
- Proposed 3-D SGFEM provides significantly better conditioning and accuracy than GFEM/XFEM
- Condition number of the SGFEM is of the same order as in the FEM
- SGFEM is more accurate than GFEM/XFEM for both geometrical and topological enrichments
- Vector-valued singular enrichments yield better conditioning than scalar-valued
- OD is only basis that can deliver optimal convergence *and* good conditioning, simultaneously

Ongoing work on the SGFEM

- Enrichments for non-planar 3-D crack surfaces
- Enrichments for highly non-convex crack fronts
- SGFEM for global-local enrichments







Questions?

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