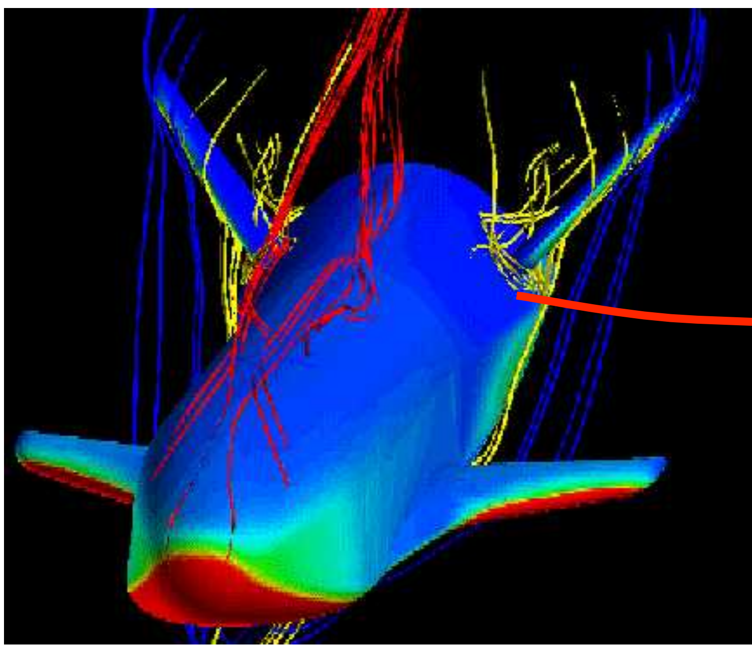
# Bridging Scales with a Generalized Finite Element method

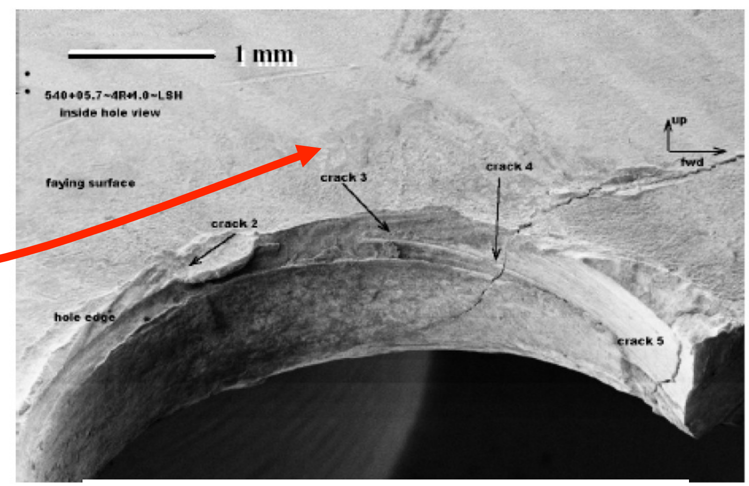
C. Armando Duarte
Dept. of Civil and Environmental Engineering
Computational Science and Engineering
University of Illinois at Urbana-Champaign, Urbana, IL

International Conference on Extended Finite Element Methods - XFEM 2011
June 29 – July 1, 2011, Cardiff, United Kingdom

# Motivation: The Need to Bridge Scales

- **Mechanically-Short Cracks**
    - Hypersonic aircrafts are subjected to intense thermo, mechanical and acoustic loads
    - Most of the life of the aircraft structure corresponds to the incubation and growth of micro-cracks



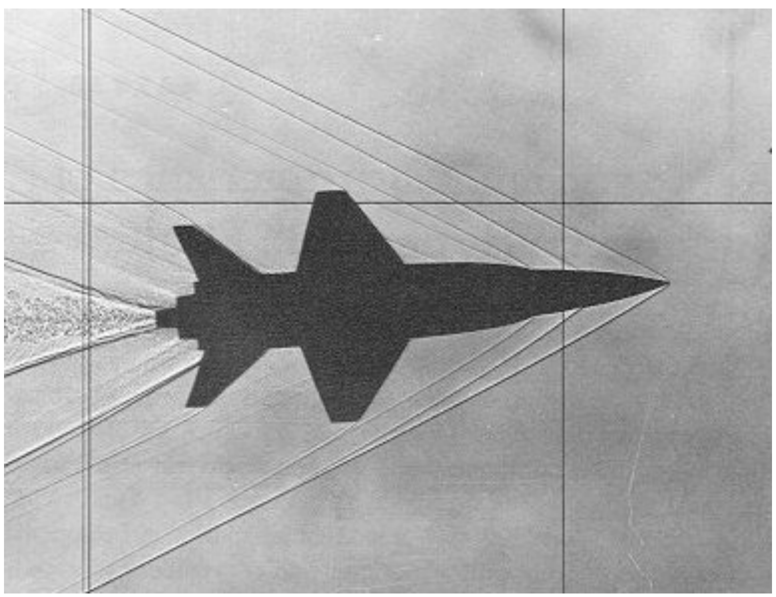


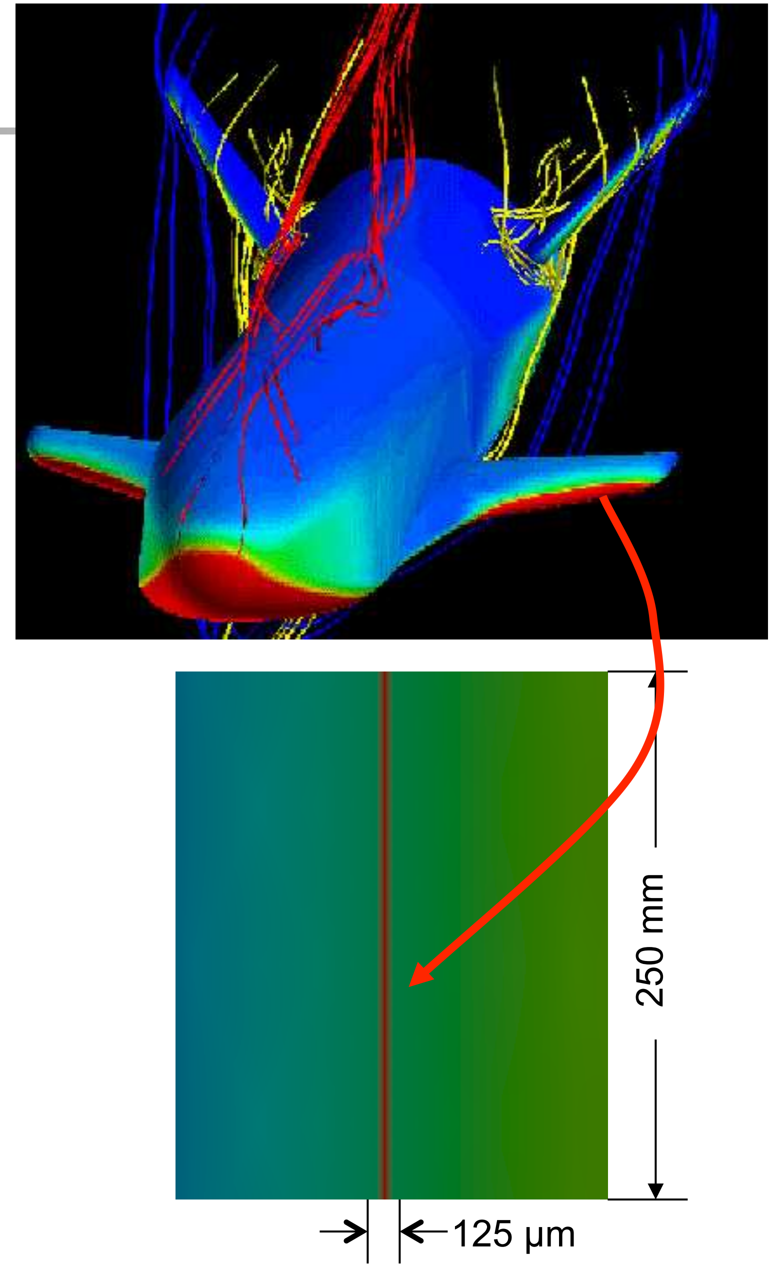

Multiple cracks around a rivet hole

[Sandia National Lab, 2005]

# Bridging Scales

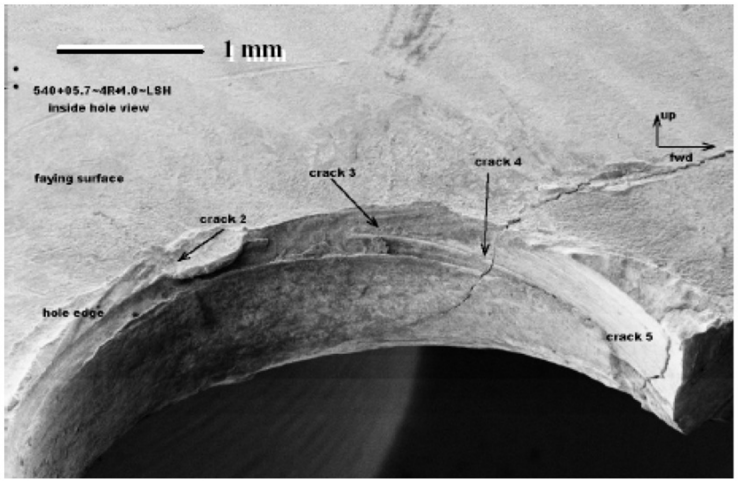
- **Thermal loads on hypersonic aircrafts**
- Shock wave impingements cause large thermal gradients
- Experiments are difficult and limited





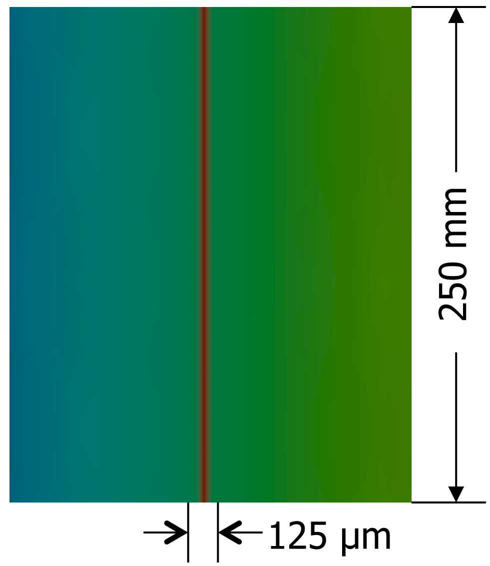

Dimensions not to scale

# Multi-scale Problems

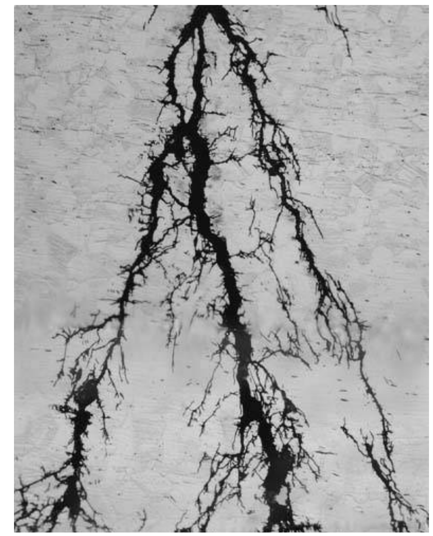



Multiple cracks around a rivet hole

[Sandia National Lab, 2005]




Thermal loads on hypersonic aircrafts
(dimensions not to scale)



Multiple interacting fractures

- Predictive simulations require modeling of phenomena spanning several spatial and temporal scales
- Advances in existing computational methods are needed
- Increasing computational power alone is not enough

# Outline

- Generalized finite element methods: Basic ideas
- Bridging scales with the GFEM:
  - Global-local enrichments
- Applications and mathematical analysis
- Transition: Non-intrusive implementation in Abaqus
- Extension to nonlinear problems
- Closing remarks

# Early works on Generalized FEMs

- Babuska, Caloz and Osborn, 1994 (Special FEM).
- Duarte and Oden, 1995 (Hp Clouds).
- Babuska and Melenk, 1995 (PUFEM).
- Oden, Duarte and Zienkiewicz, 1996 (Hp Clouds/GFEM).
- Duarte, Babuska and Oden, 1998 (GFEM).
- Belytschko et al., 1999 (Extended FEM).
- Strouboulis, Babuska and Copps, 2000 (GFEM).

- Basic idea:
  - Use a partition of unity to build Finite Element shape functions

- Recent review papers

  Belytschko T., Gracie R. and Ventura G. A review of extended/generalized finite element methods for material modeling, *Mod. Simul. Matl. Sci. Eng.,* 2009

  Fries, T.-P. and Belytschko, T. The generalized/extended finite element method: An overview of the method and its applications, *Int. J. Num. Meth. Eng.*, 2010.

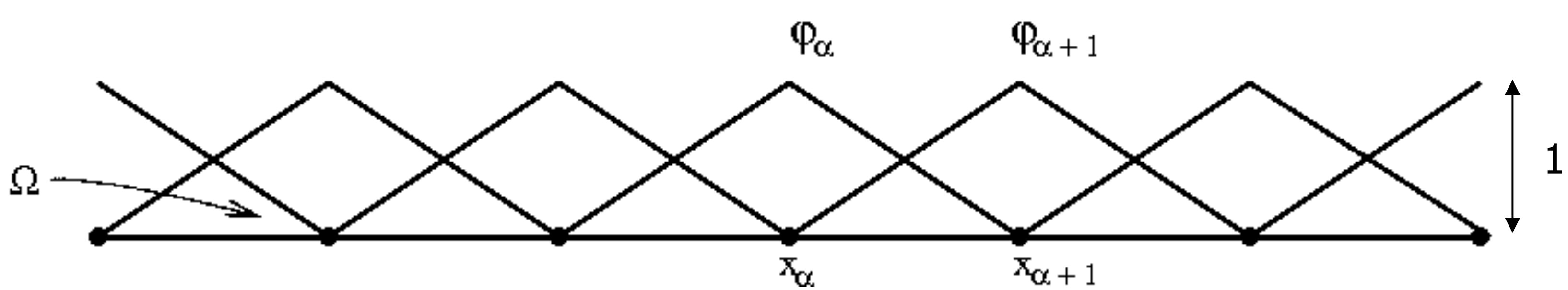
# Generalized Finite Element Method

GFEM can be interpreted as a FEM with shape functions built using the concept of a partition of unity

*Partition of Unity* (PoU)

$$\boxed{\sum_{\alpha} \varphi_{\alpha}(x) = 1 \qquad \forall \; x \in \Omega}$$

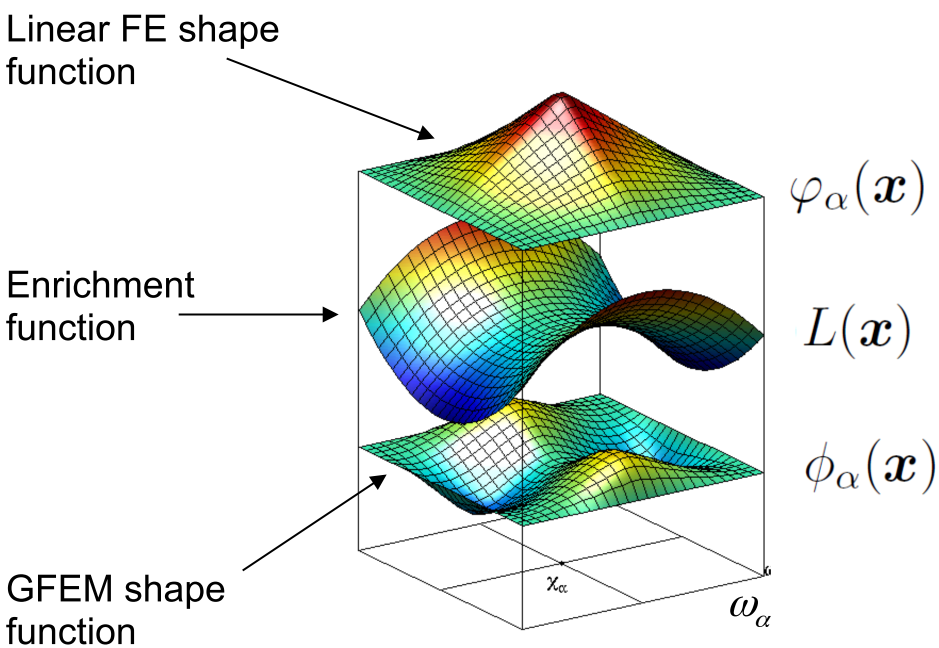
- $\varphi_{\alpha}$ = Linear FEM shape function



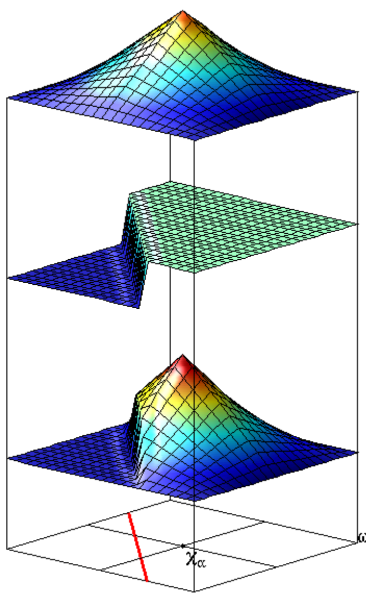
# Generalized Finite Element Method

- GFEM shape function = FE shape function * enrichment function

$$\boxed{\phi_\alpha(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})L(\boldsymbol{x})}$$

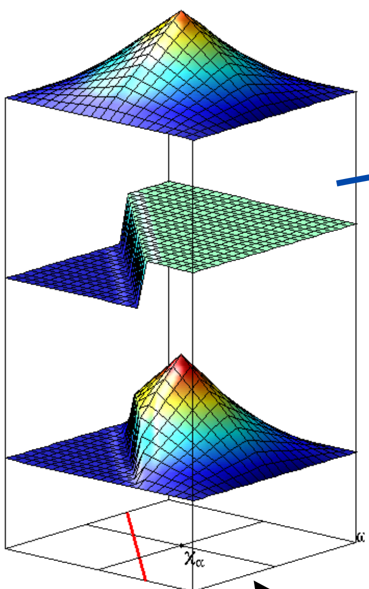



- Allows construction of shape functions incorporating a-priori knowledge about solution



# GFEM Approximation for 3-D Cracks

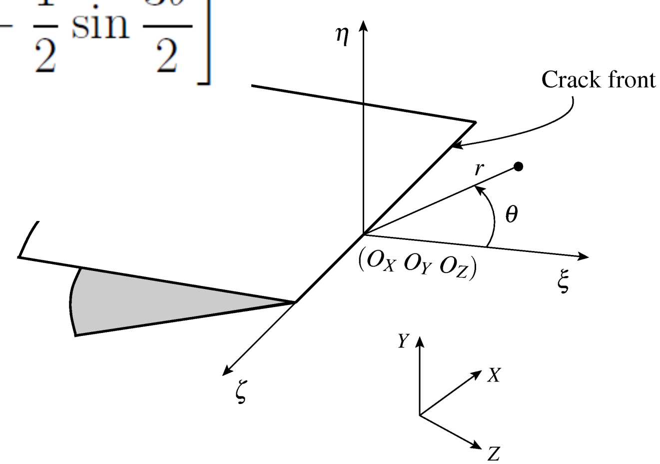
$$\boldsymbol{X}^{hp}(\Omega) = \left\{ \boldsymbol{u} = \sum_{\alpha=1}^{N} \underbrace{\varphi_\alpha(\boldsymbol{x})}_{\text{PoU}} \left[ \underbrace{\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{polynomial}} + \underbrace{\mathcal{H}\tilde{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{discontinuous}} + \underbrace{\breve{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{singular}} \right] \right\}$$



$$\begin{aligned}
\breve{L}^{\xi}_{\alpha 1}(r,\theta) &= \sqrt{r}\left[(\kappa - \frac{1}{2})\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right] \\
\breve{L}^{\eta}_{\alpha 1}(r,\theta) &= \sqrt{r}\left[(\kappa + \frac{1}{2})\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2}\right] \\
\breve{L}^{\zeta}_{\alpha 1}(r,\theta) &= \sqrt{r}\sin\frac{\theta}{2}
\end{aligned}$$

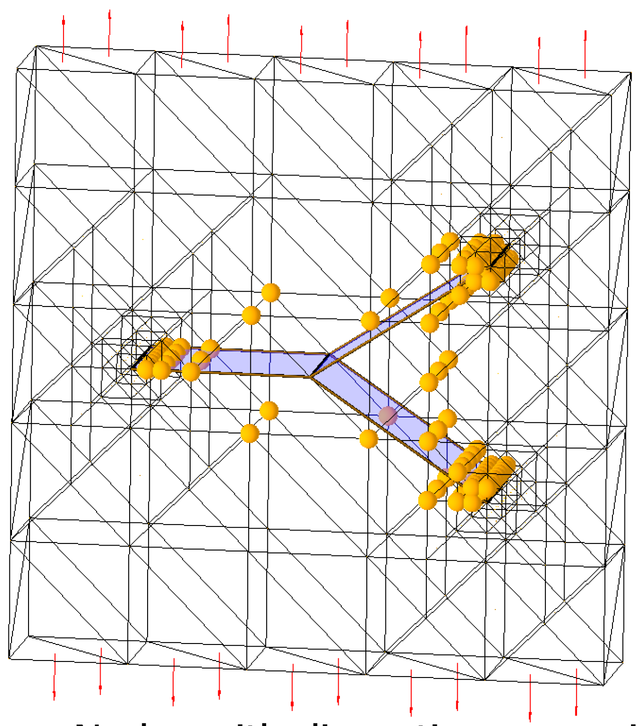
[Duarte and Oden 1996]

cloud or patch $\alpha$

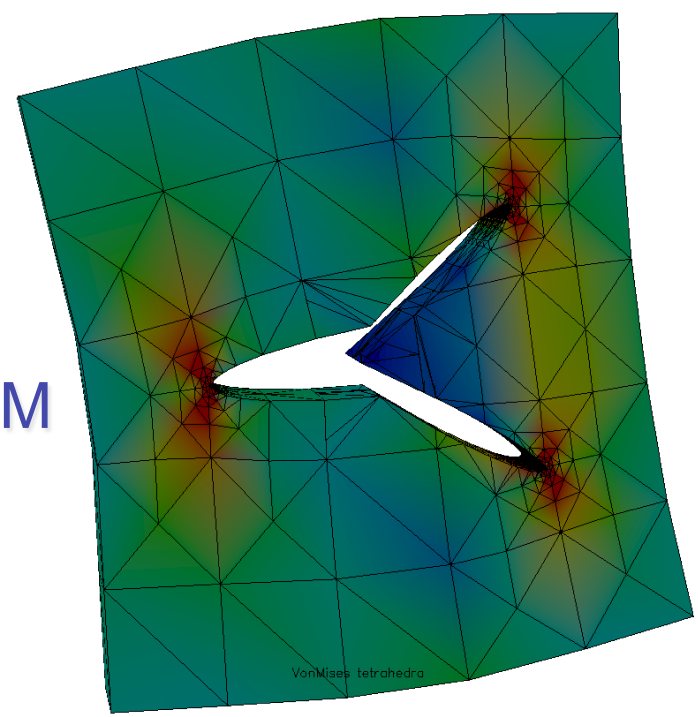


# Modeling Cracks with hp-GFEM

- Discontinuities modeled via enrichment functions, *not* the FEM mesh
- Mesh refinement *still required* for acceptable accuracy



*hp*-GFEM



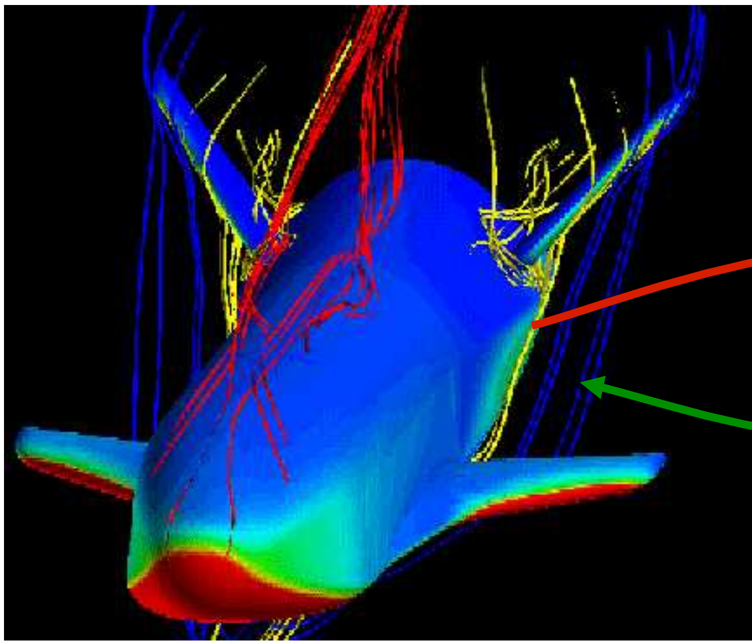

● = Nodes with discontinuous enrichments

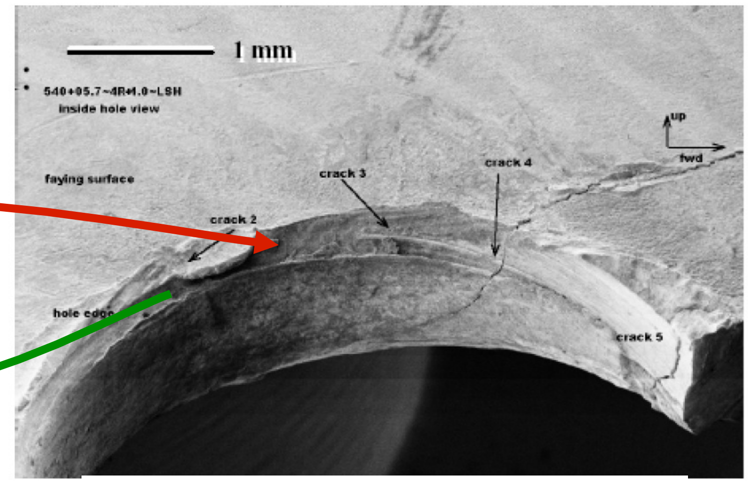
Von Mises stress

[Duarte et al., Int. J. Num. Meth. Eng., 2007]

# Bridging Scales with Global-Local Enrichment Functions

- How to account for interactions among scales?






Multiple cracks around a rivet hole

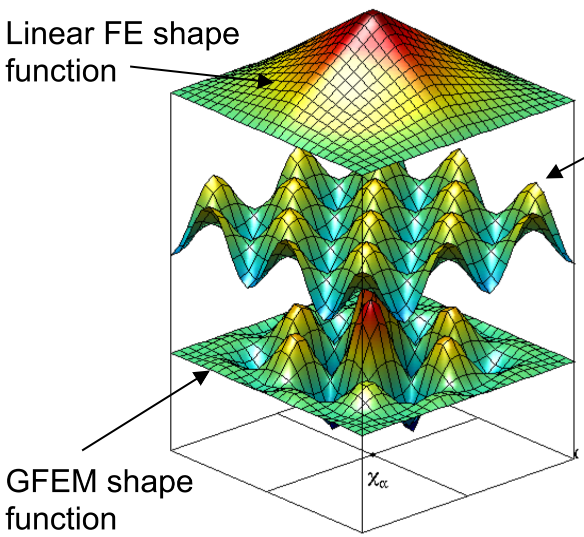
[Sandia National Lab, 2005]

## Goal:

- Capture fine scale effects on *coarse* meshes at the global (structural) scale

# Bridging Scales with Global-Local Enrichment Functions *

- Enrichment functions computed from solution of local boundary value problems: Global-Local enrichment functions



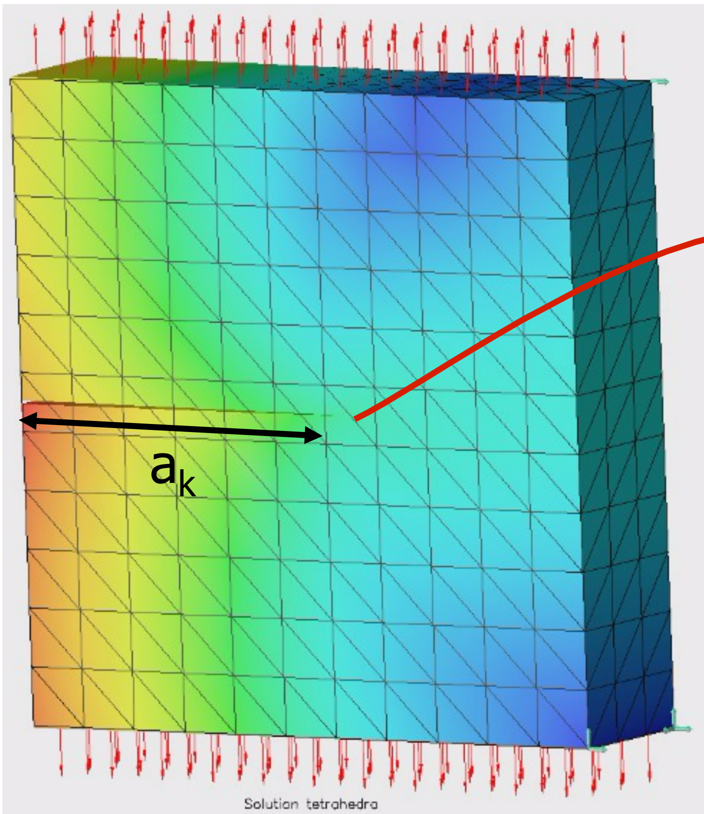

Enrichment = Numerical solutions of BVP

- **Idea:** *Use available numerical solution at a simulation step to build shape functions for next step* (quasi-static, transient, non-linear, etc.)
- Enrichment functions are produced numerically on-the-fly through a global-local analysis
- Use a *coarse* mesh enriched with Global-Local (G-L) functions

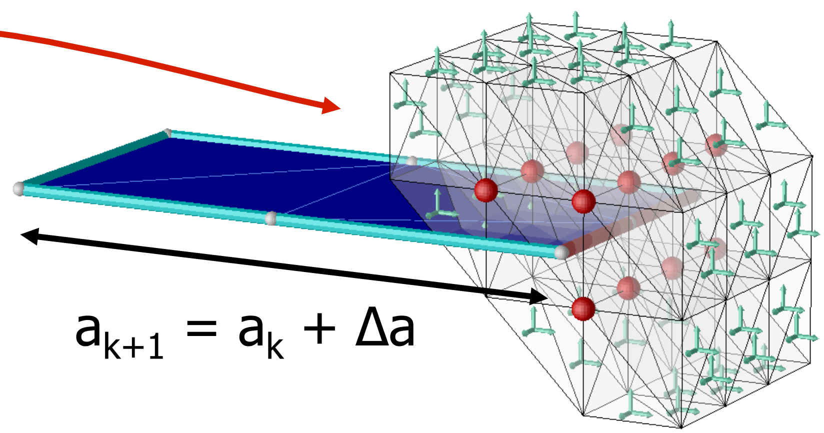
* Duarte et al. 2005, 2007, 2008, 2010, 2011

# Global-Local Enrichments for 3-D Fractures

- $u_G^k$ solution of global problem at crack step k




- Define local domain containing crack front at step k+1

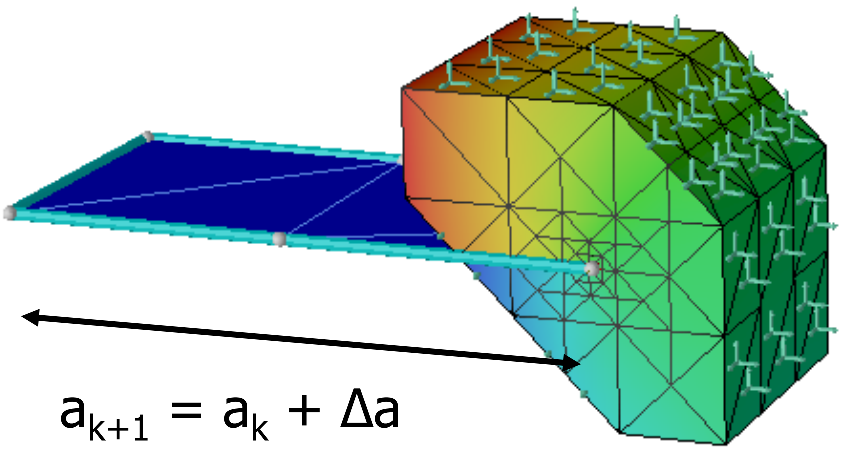



Local problem with crack size $a_{k+1}$

$u_G^k \in X_G^k(\Omega)$ = solution of global problem with crack size $a_k$

# Global-Local Enrichments for 3-D Fractures

- Solve local problem at step k using *hp*-GFEM




Boundary conditions for local problems provided by global solution:

$$\boldsymbol{u}_L^k = \boldsymbol{u}_G^k \quad \text{on} \quad \partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)$$

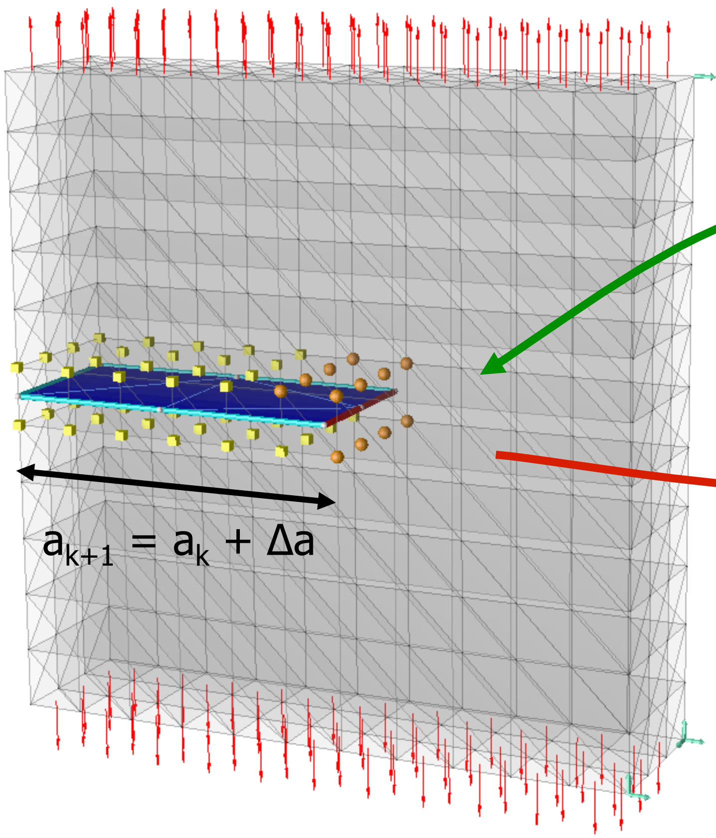
$\boldsymbol{X}_L^k\left(\Omega_L^k\right)$ = *hp*-GFEM space

Find $\boldsymbol{u}_L^k \in \boldsymbol{X}_L^k\left(\Omega_L^k\right) \subset H^1\left(\Omega_L^k\right)$ such that $\forall\, \boldsymbol{v}_L^k \in \boldsymbol{X}_L^k\left(\Omega_L^k\right)$

$$\int_{\Omega_L^k} \boldsymbol{\sigma}(\boldsymbol{u}_L^k) : \boldsymbol{\varepsilon}(\boldsymbol{v}_L^k) d\boldsymbol{x} + \kappa \int_{\partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)} \boldsymbol{u}_L^k \cdot \boldsymbol{v}_L^k ds$$
$$= \int_{\partial\Omega_L^k \cap \partial\Omega^\sigma} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}_L^k ds + \kappa \int_{\partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)} \boldsymbol{u}_G^k \cdot \boldsymbol{v}_L^k ds$$

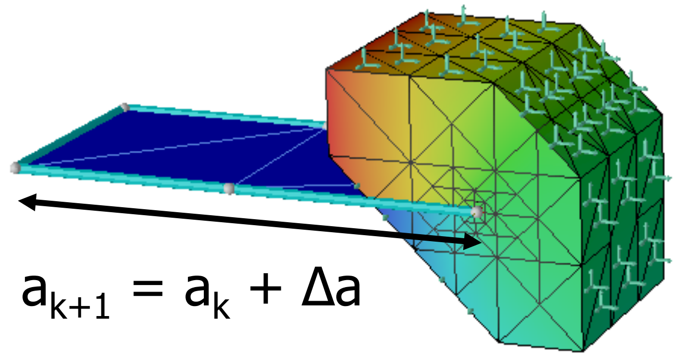
# Global-Local Enrichments for 3-D Fractures

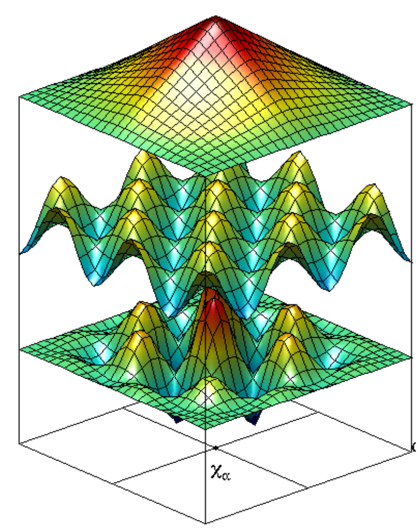
- **Defining Step:** Global space is enriched with local solutions




global-local enrichments

BC for local problem



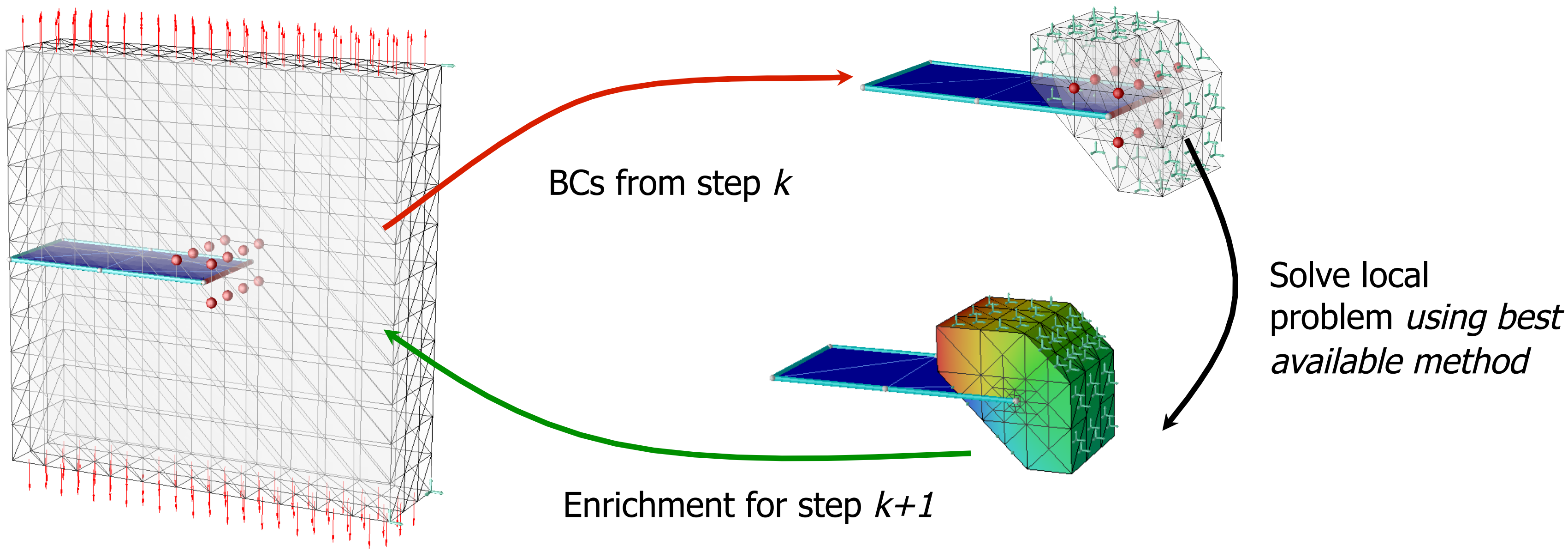

$$\boxed{\phi_\alpha^{k+1}(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})\boldsymbol{u}_L^k(\boldsymbol{x})}$$

$\boldsymbol{u}_G^{k+1} \in \boldsymbol{X}_G^{k+1}(\Omega)$ = solution of global problem with crack size $a_{k+1}$

- Procedure may be repeated: Update local BCs and enrichment functions

# Global-Local Enrichments for Crack Growth

- **Summary:** Use solution of global problem at simulation *k* to build enrichment functions for step *k+1*



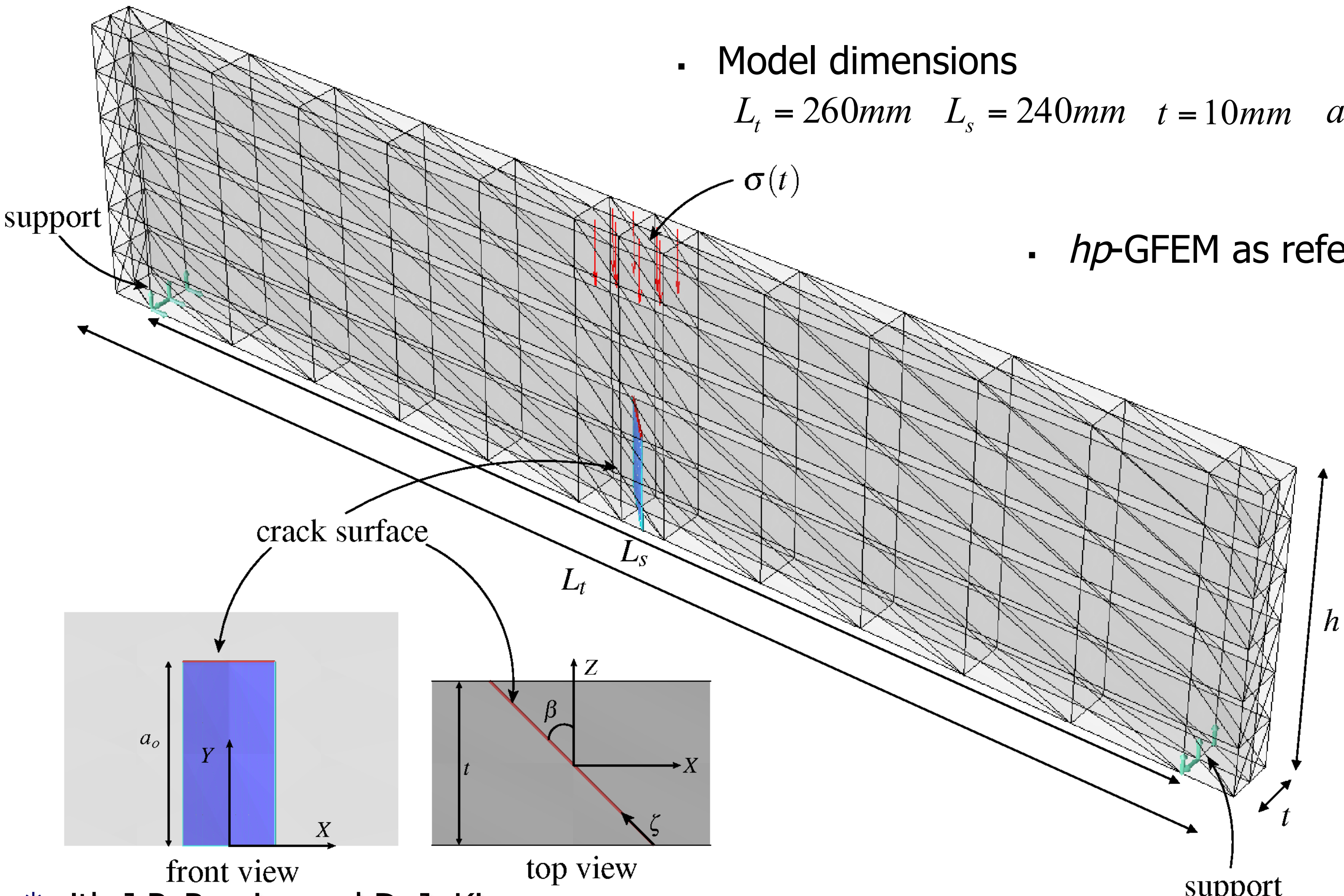

- Discretization spaces updated on-the-fly with global-local enrichment functions

$$\boldsymbol{X}_G^{k+1}(\Omega_G) = \left\{ \boldsymbol{u} = \underbrace{\sum_{\alpha=1}^{N} \varphi_\alpha(\boldsymbol{x})\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{coarse-scale approx.}} + \underbrace{\sum_{\beta\in\mathcal{I}_{gl}^k} \varphi_\beta(\boldsymbol{x})\boldsymbol{u}_\beta^{gl(k)}(\boldsymbol{x})}_{\text{fine-scale approx.}} \right\}$$

$\boldsymbol{u}_\beta^{gl(k)}$ = G-L enrichment

# Edge-Notched Beam with Slanted Crack *

- **Fatigue Crack Growth: hp-GFEM and GFEM$^{gl}$ solutions**




- Model dimensions

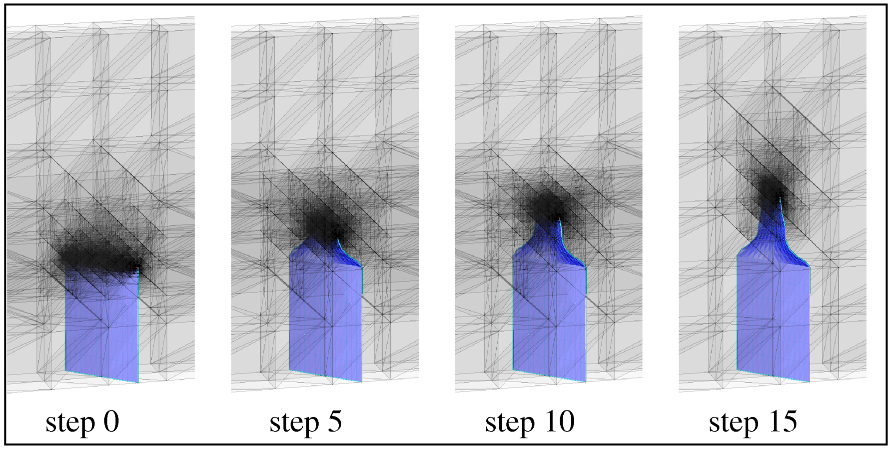
  $L_t = 260mm \quad L_s = 240mm \quad t = 10mm \quad a_o/h = 1/3 \quad \beta = 45^o$

- *hp*-GFEM as reference solution

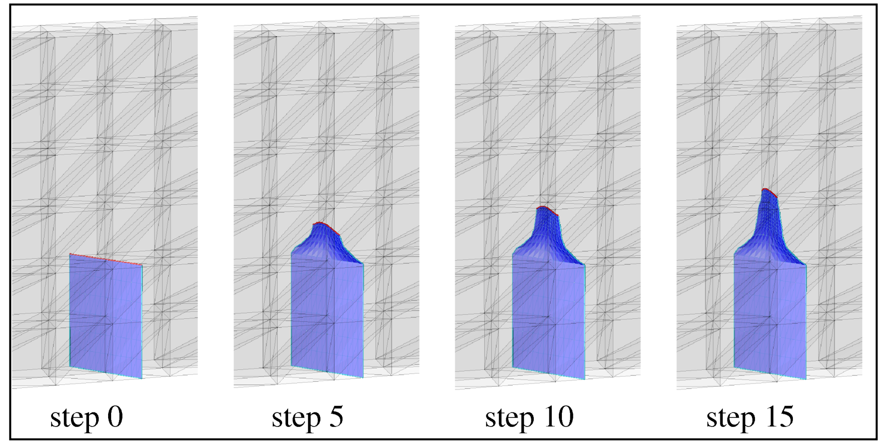
Movie

*with J.P. Pereira and D.-J. Kim

# Edge-Notched Beam with Slanted Crack



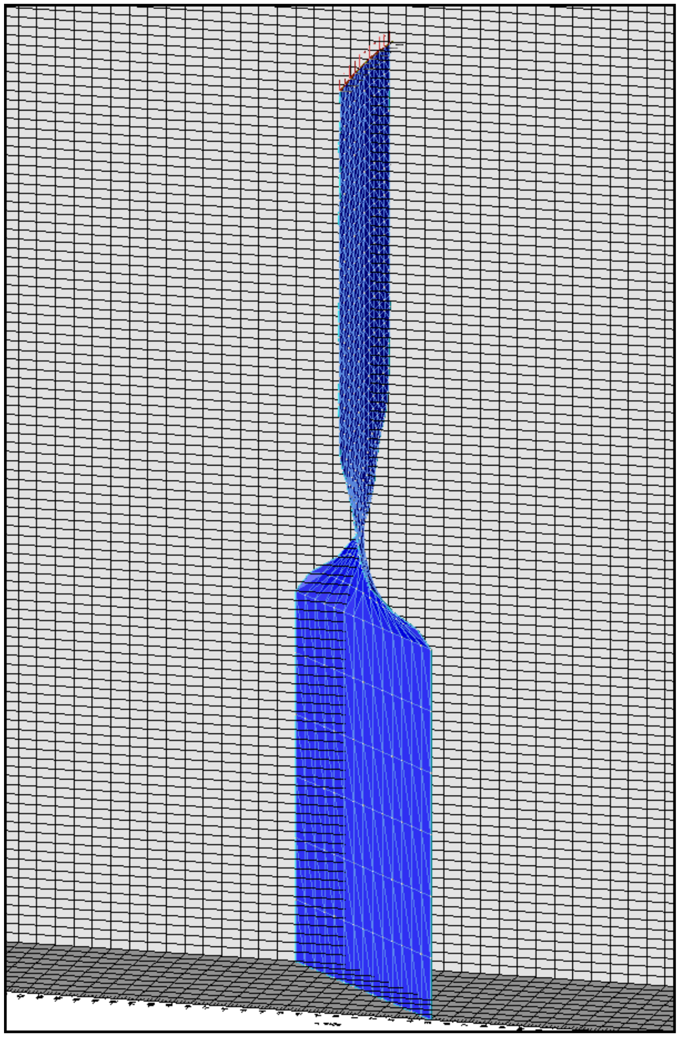

Available Methods – *hp*-GFEM/FEM

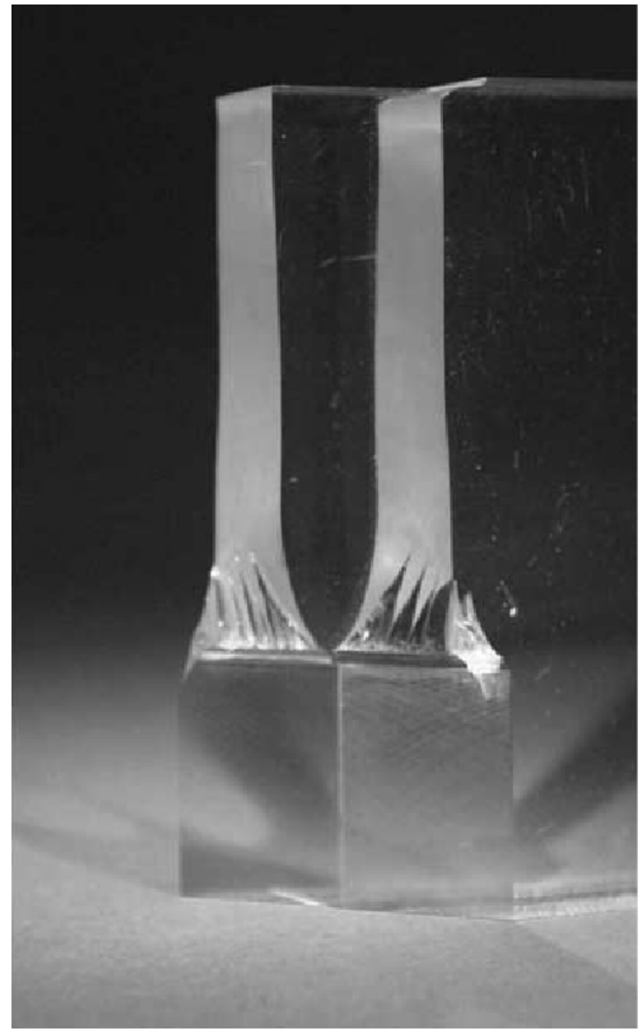



Two-Scale Generalized FEM – $GFEM^{gl}$

- Mesh with elements that are orders of magnitude larger than in a FEM mesh
- Fully compatible with FEM
- Single field formulation: Does not introduce stability (LBB) issues

# Experimental Results

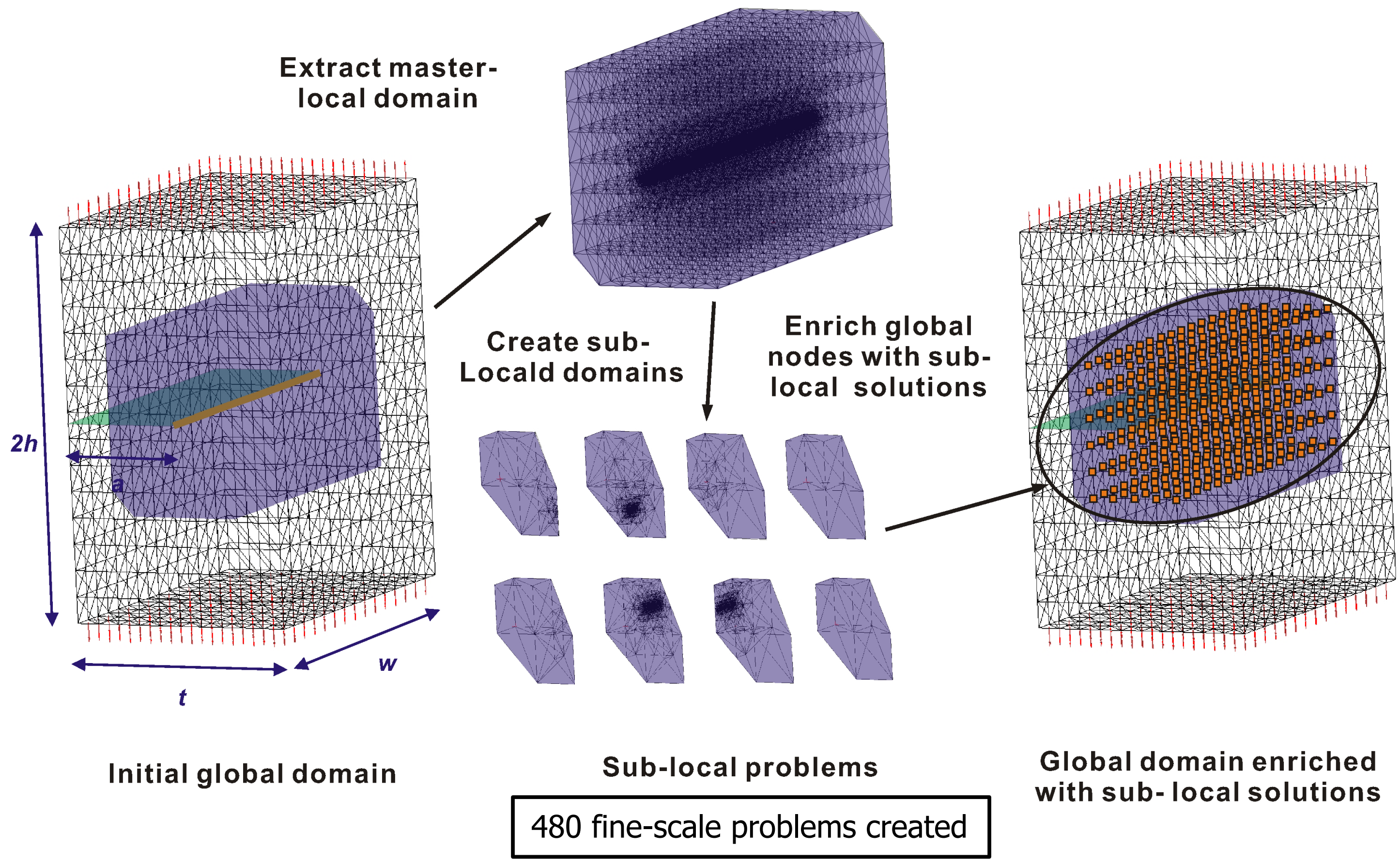




[Buchholz et al., 2004]

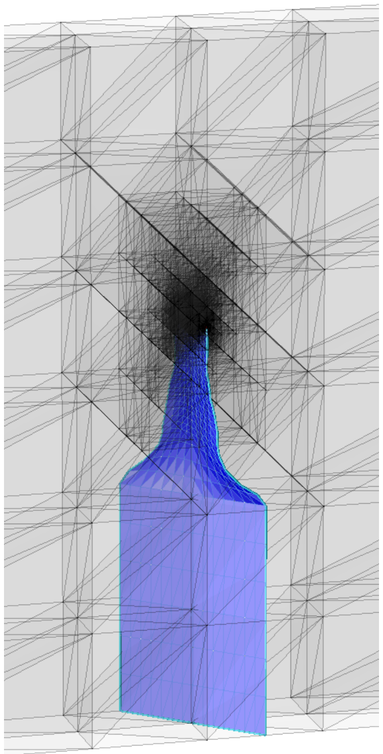
# Parallel Computation of Enrichment Functions *

- A large number of small fine-scale problems can be created instead of a single one
- *No communication is involved in their parallel solution*

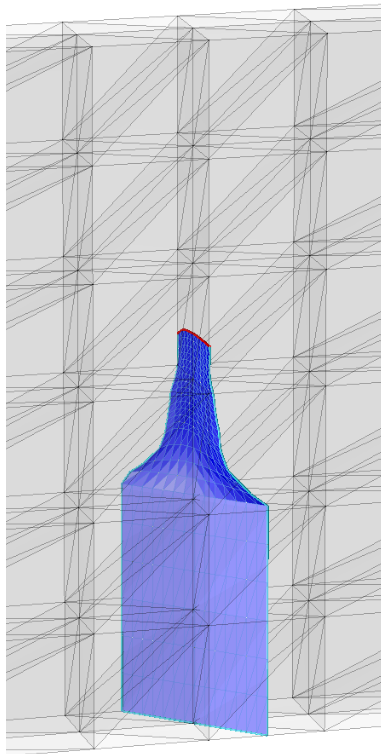



*with D.-J. Kim and N. Sohb

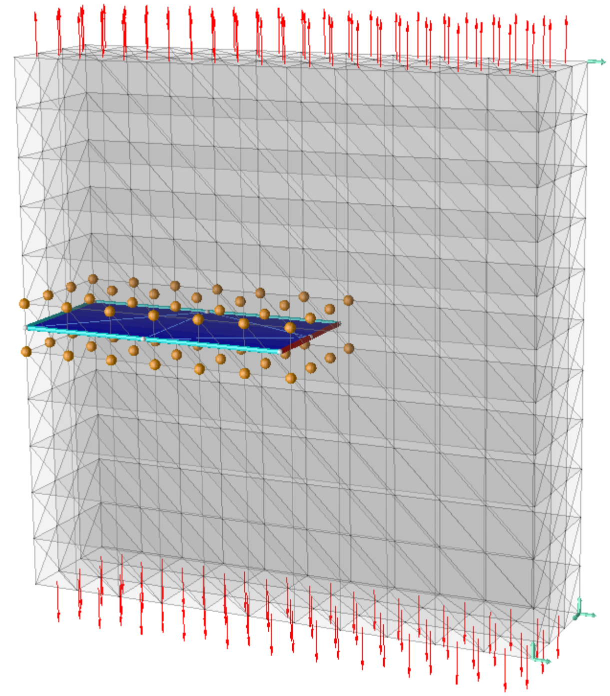
# Mathematical Analysis *



hp-GFEM/FEM



GFEM$^{gl}$



GFEM$^{gl}$: Error controlled through global-local enrichments

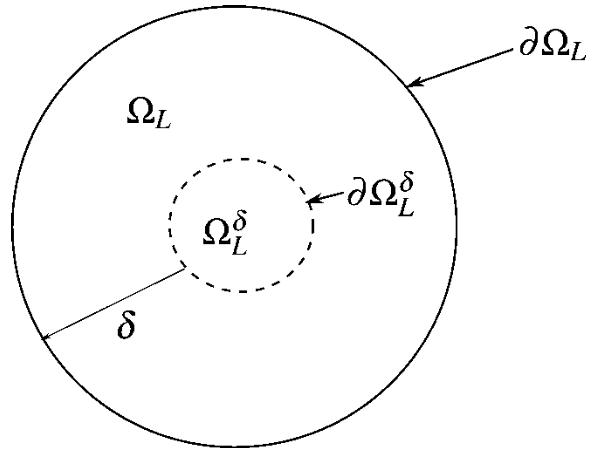
**Questions:**

- What are the effects of inexact BCs at fine-scale problems?
- How to control them?

*with V. Gupta

# A-Priori Error Estimate

- Local error estimate



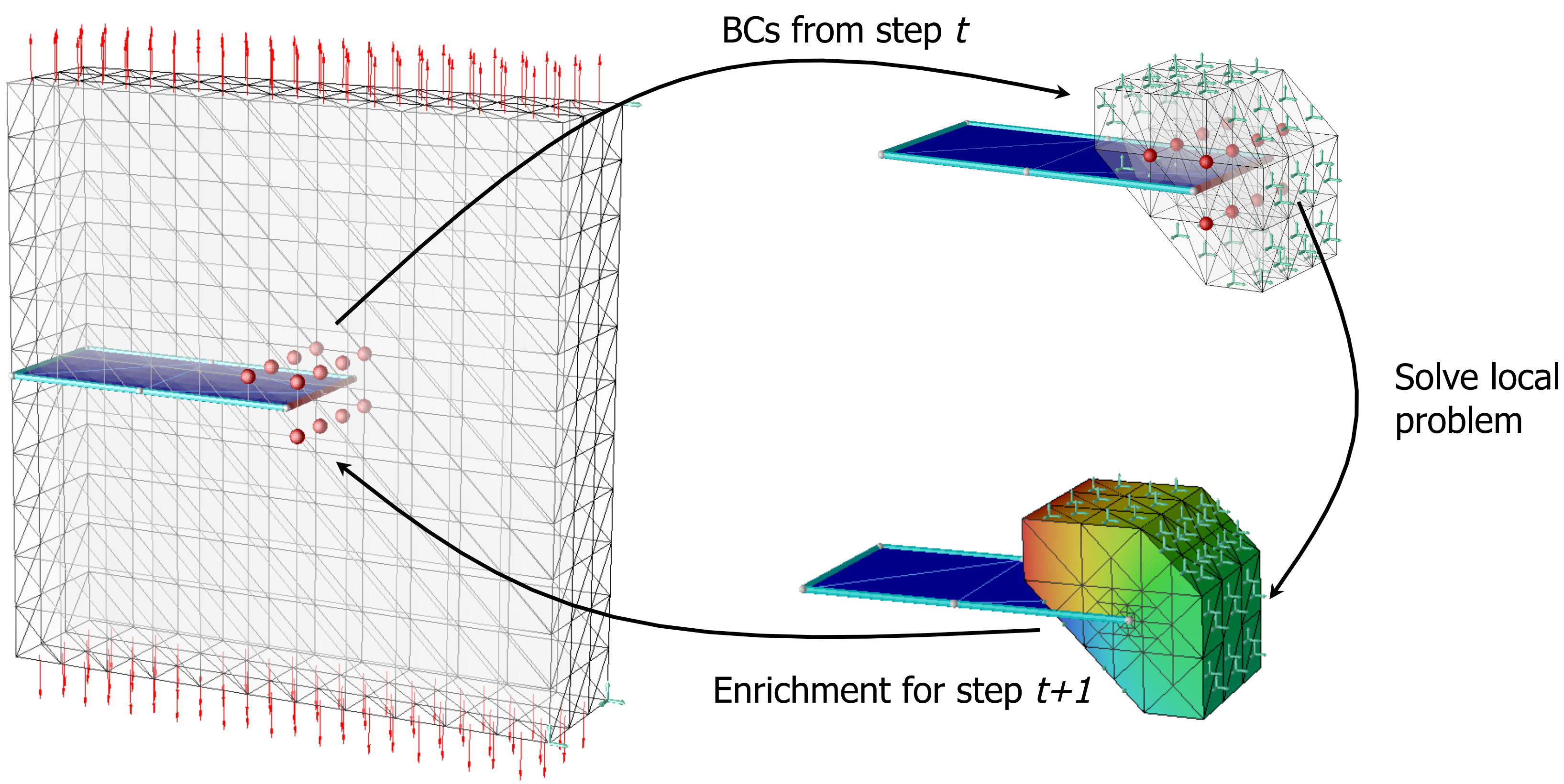

$$\|\boldsymbol{u}^{exBC} - \boldsymbol{u}_h^{inexBC}\|_{\varepsilon(\Omega_L^\delta)} \leq C \underbrace{\inf_{\boldsymbol{x}\in \boldsymbol{X}_L^{hp}(\Omega_L)} \|\boldsymbol{u}^{inexBC} - \boldsymbol{x}\|_{\varepsilon(\Omega_L)}}_{\text{Discretization error}} + \underbrace{\frac{C_1}{\delta}\|\boldsymbol{u}^{exBC} - \boldsymbol{u}^{inexBC}\|_{L^2(\Omega_L)}}_{\text{Effect of inexact BC}}$$

- Global Error [Babuska and Melenk, 1996]

$$\|\boldsymbol{u}-\boldsymbol{u}_G\|^2_{\varepsilon(\Omega)} \leq C \sum_{\alpha=1}^{N} \inf_{\boldsymbol{u}_\alpha \in \chi_\alpha} \|\boldsymbol{u}-\boldsymbol{u}_\alpha\|^2_{\varepsilon(\omega_\alpha)} \leq C \sum_{\alpha=1}^{N} \|\boldsymbol{u}-\boldsymbol{u}_h^{\text{inexBC}}\|^2_{\varepsilon(\omega_\alpha)}$$

where $\boldsymbol{u} \equiv \boldsymbol{u}^{\text{exBC}}$

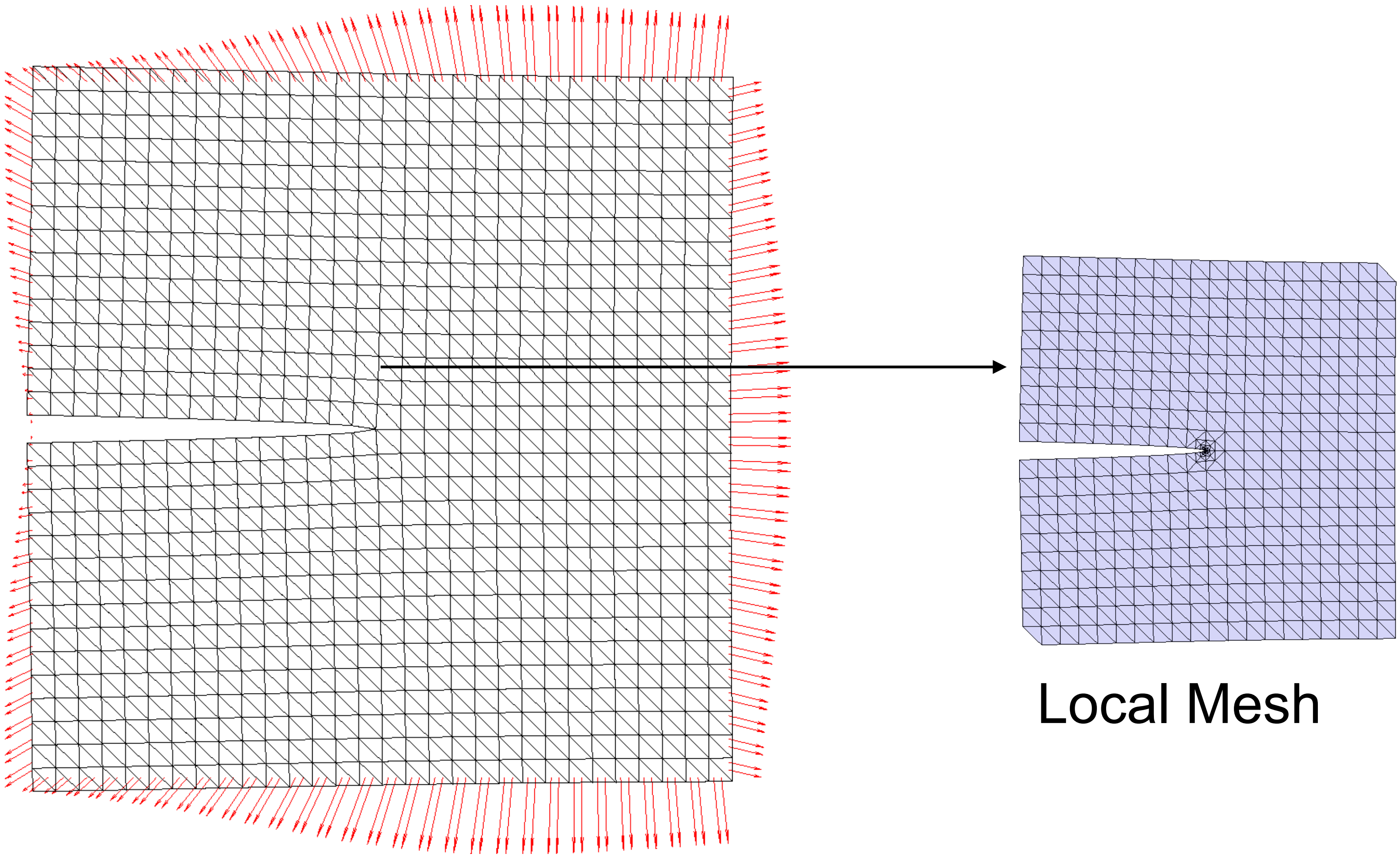
# Strategy I: Multiple Global-Local Iterations




- Repeat Global-local-Global cycle before advancing crack

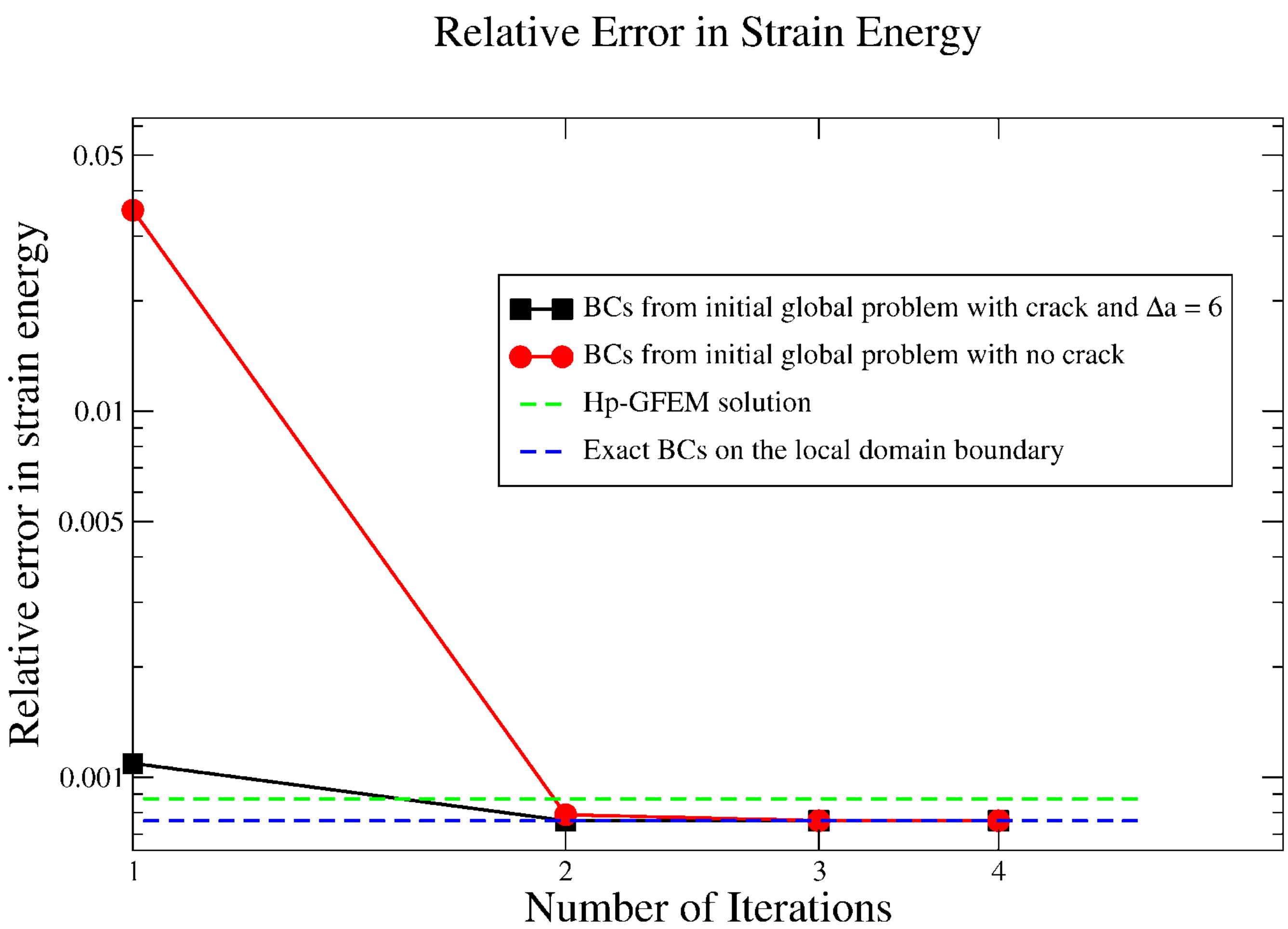
# Strategy I: Multiple Global-Local Iterations

- 30" x 30" x 1" edge-crack panel loaded with Mode I tractions



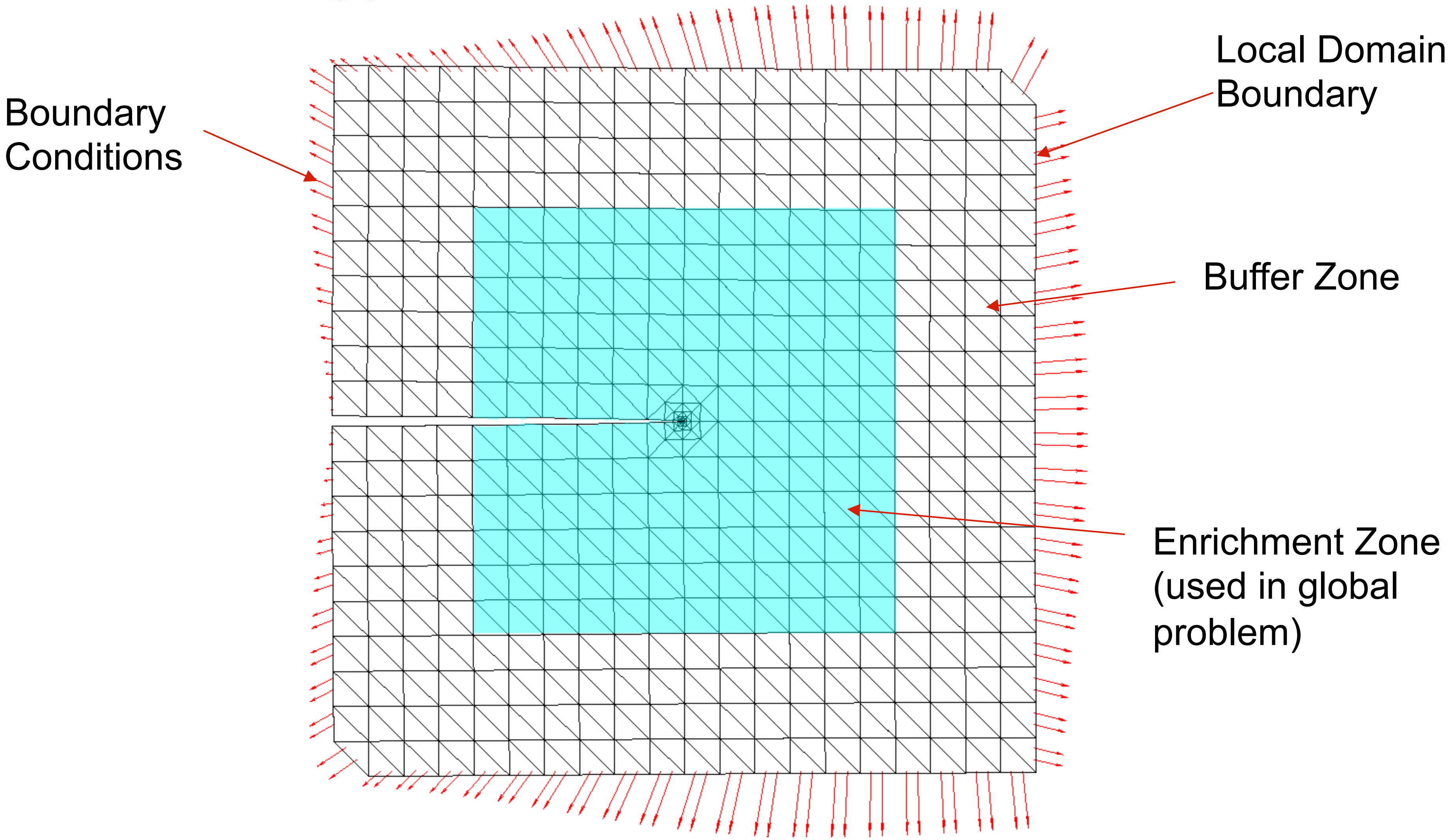
Local Mesh

# Strategy I: Multiple Global-Local Iterations



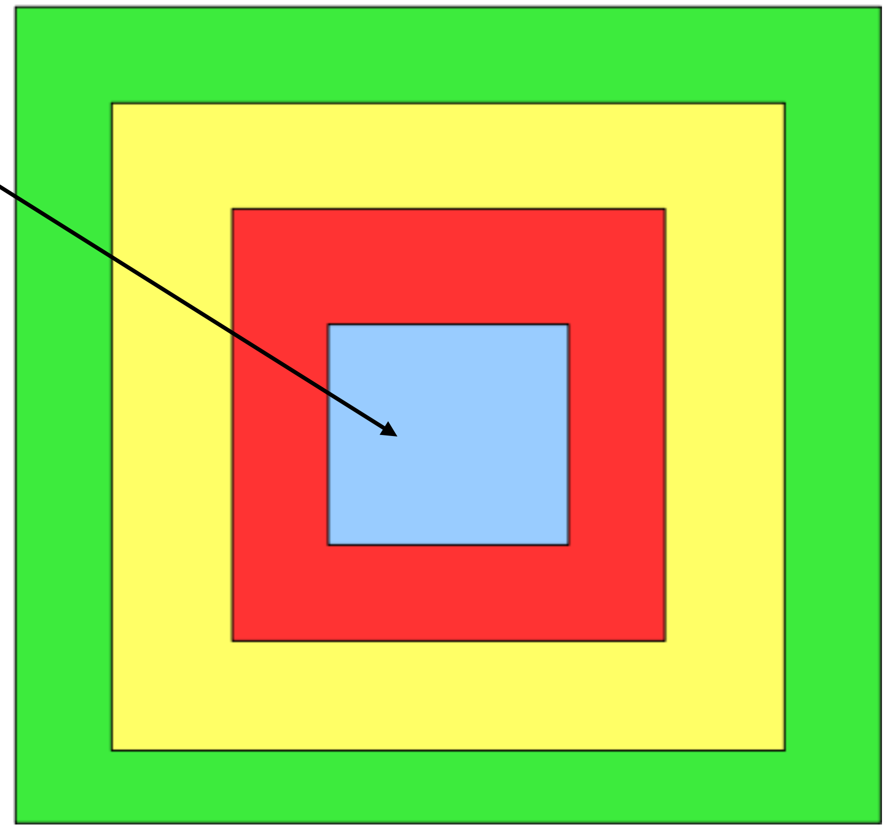

- GFEM$^{gl}$ can deliver same accuracy as hp-GFEM (DNS)

# Strategy II: Buffer Zone in Local Domain



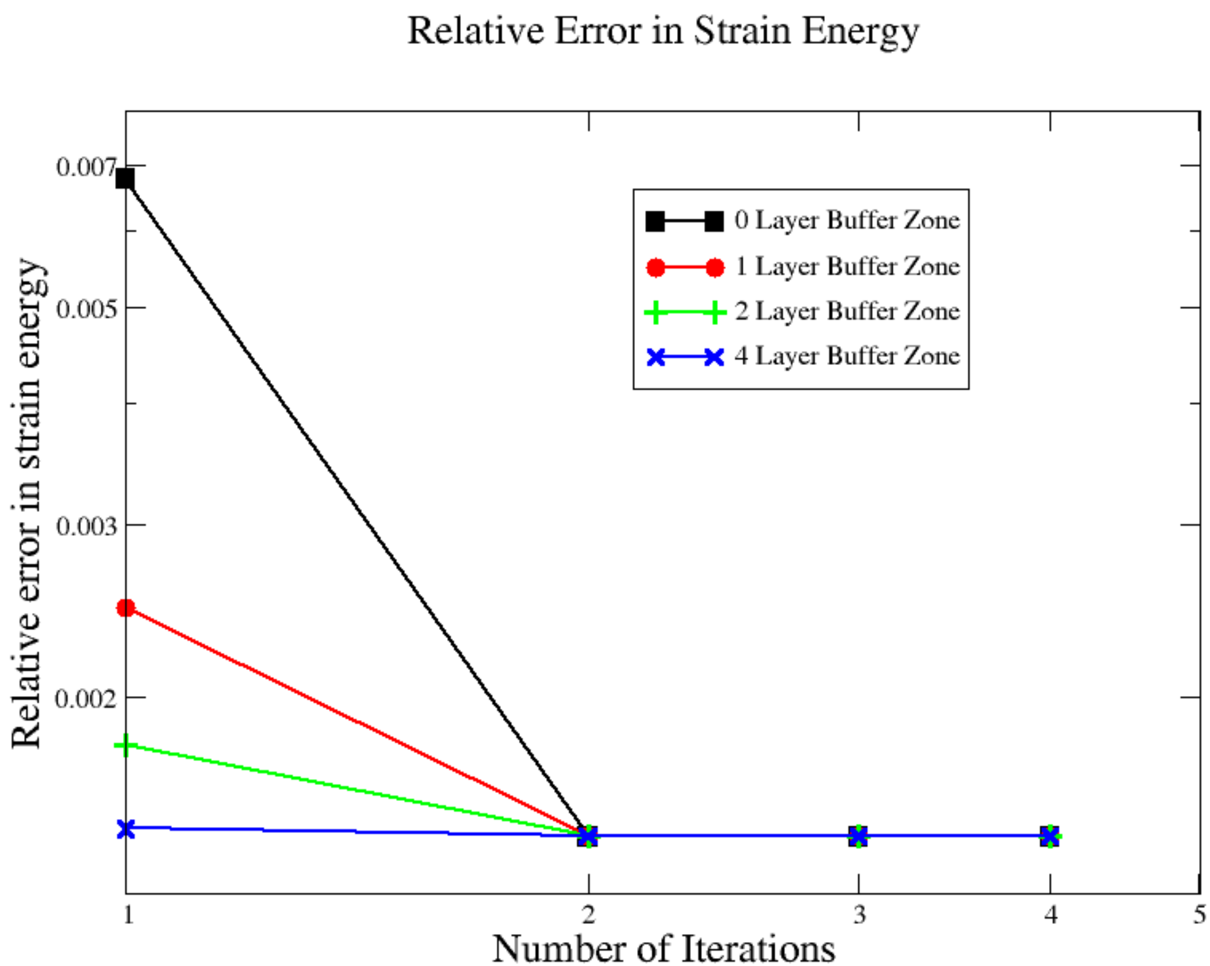
# Strategy II: Buffer Zone in Local Domain

- Buffer Zone Sizes Considered
  - Enrichment Zone: 4” X 4” blue square region
  - Buffer zone (in terms of number of layers of elements):
    - Red - 1 layer
    - Yellow - 2 layers
    - Green - 4 layers



Not to scale

# Strategy II: Buffer Zone in Local Domain




- BCs from global problem *without* a crack

# Outline

- Generalized finite element methods: Basic ideas
- Bridging scales with the GFEM:
  - Global-local enrichments
- Applications and mathematical analysis
- Transition: Non-intrusive implementation in Abaqus
- Extension to nonlinear problems
- Closing remarks

# Computation of Solution at a Crack Step

$$\boldsymbol{u}_G = \underbrace{\tilde{\boldsymbol{u}}^0}_{\text{coarse scale (polynomial)}} + \underbrace{\boldsymbol{u}^{\text{gl}}}_{\text{fine scale (G-L)}} = [\boldsymbol{N}^0 \boldsymbol{N}^{\text{gl}}] \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^0 \\ \underline{\boldsymbol{u}}^{\text{gl}} \end{bmatrix}$$

$$\begin{aligned} \underline{\tilde{\boldsymbol{u}}}^0 &= \text{DOFs associate with coarse scale discretization} \\ \underline{\boldsymbol{u}}^{\text{gl}} &= \text{DOFs associate with G-L (hierarchical) enrichments} \end{aligned}$$

$$\text{dim}(\underline{\boldsymbol{u}}^{\text{gl}}) << \text{dim}(\underline{\tilde{\boldsymbol{u}}}^0)$$

This leads to

Computed by FEM code

$$\begin{bmatrix} \boldsymbol{K}^0 & \boldsymbol{K}^{0,\text{gl}} \\ \boldsymbol{K}^{\text{gl},0} & \boldsymbol{K}^{\text{gl}} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^0 \\ \underline{\boldsymbol{u}}^{\text{gl}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^0 \\ \boldsymbol{F}^{\text{gl}} \end{bmatrix}$$

Solve using, e.g., static condensation of $\underline{\boldsymbol{u}}^{\text{gl}}$

# Computation of Solution at a Crack Step

From the first equation

$$
\begin{aligned}
\underline{\tilde{\boldsymbol{u}}}^{\,0} &= (\boldsymbol{K}^0)^{-1}\boldsymbol{F}^0 - (\boldsymbol{K}^0)^{-1}\boldsymbol{K}^{0,\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} \\
&= \underline{\boldsymbol{u}}^{\,0} - \boldsymbol{S}^{0,\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}}
\end{aligned}
$$

Where

$$\boldsymbol{S}^{0,\mathrm{gl}} := (\boldsymbol{K}^0)^{-1}\boldsymbol{K}^{0,\mathrm{gl}}$$

$$\boxed{\boldsymbol{K}^0 \underbrace{\boldsymbol{S}^{0,\mathrm{gl}}}_{\text{pseudo coarse scale solutions}} = \underbrace{\boldsymbol{K}^{0,\mathrm{gl}}}_{\text{pseudo coarse scale loads}}}$$

$\boldsymbol{S}^{0,\mathrm{gl}}$ = Pseudo coarse scale solutions computed through forward and backward substitutions on $\boldsymbol{K}^0$ (by FEM code)

# Computation of Solution at a Crack Step

From the second equation and the above

$$\boldsymbol{K}^{\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} \;=\; \boldsymbol{F}^{\mathrm{gl}} - \boldsymbol{K}^{\mathrm{gl},0}\left[\underline{\boldsymbol{u}}^{\,0} - \boldsymbol{S}^{0,\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}}\right]$$

Thus

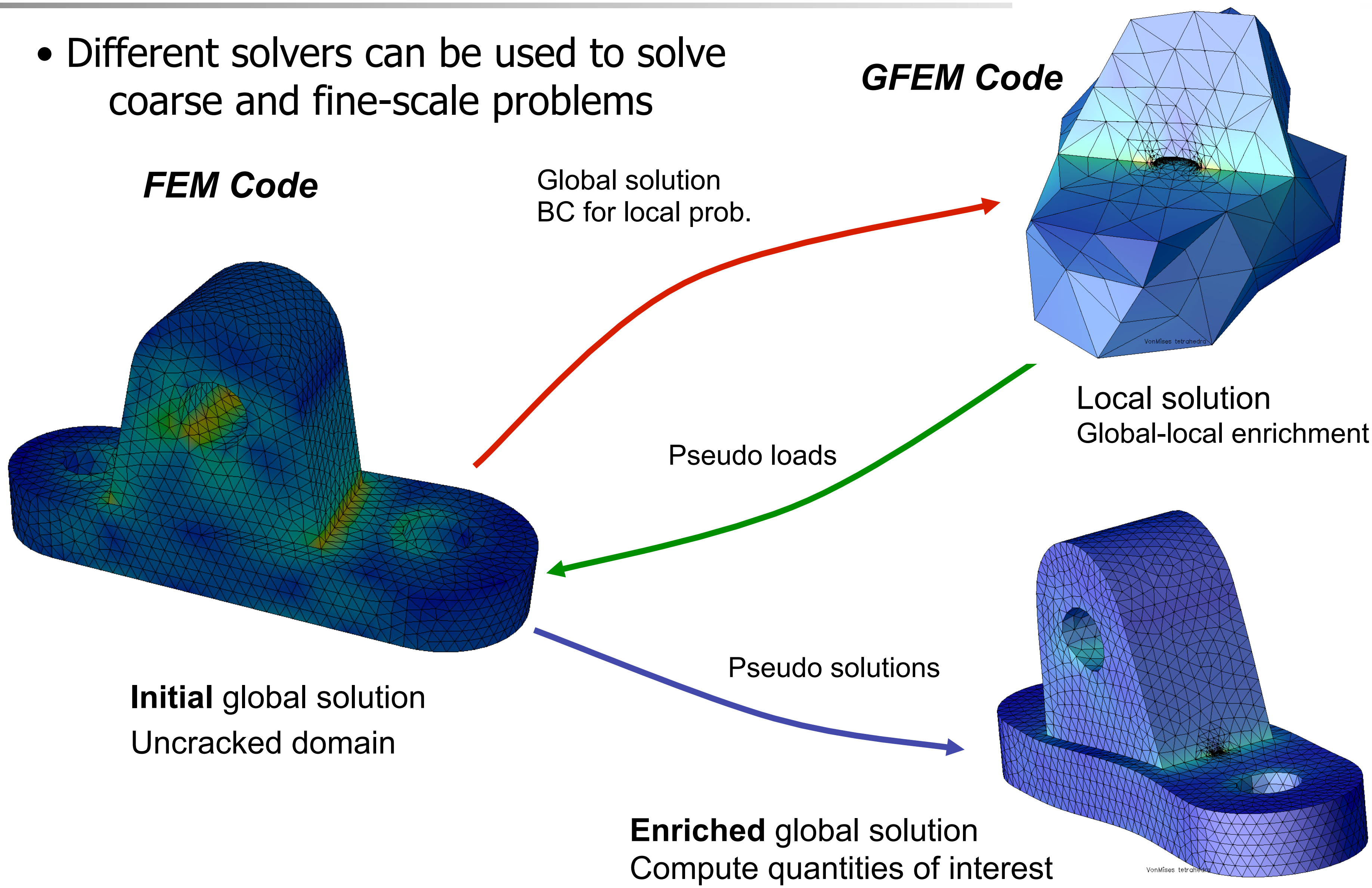
$$\underbrace{\left[\boldsymbol{K}^{\mathrm{gl}} - \boldsymbol{K}^{\mathrm{gl},0}\boldsymbol{S}^{0,\mathrm{gl}}\right]}_{\widehat{\boldsymbol{K}}^{\mathrm{gl}}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} = \underbrace{\boldsymbol{F}^{\mathrm{gl}} - \boldsymbol{K}^{\mathrm{gl},0}\,\underline{\boldsymbol{u}}^{\,0}}_{\widehat{\boldsymbol{F}}^{\mathrm{gl}}}$$

$$\boxed{\begin{gathered}
\widehat{\boldsymbol{K}}^{\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} = \widehat{\boldsymbol{F}}^{\mathrm{gl}} \\
\underline{\tilde{\boldsymbol{u}}}^{\,0} = \underline{\boldsymbol{u}}^{\,0} - \boldsymbol{S}^{0,\mathrm{gl}}\,\underline{\boldsymbol{u}}^{\,\mathrm{gl}} \\
\boldsymbol{u}_E = \tilde{\boldsymbol{u}}^{0} + \boldsymbol{u}^{\mathrm{gl}} = \left[\boldsymbol{N}^{0}\boldsymbol{N}^{\mathrm{gl}}\right]\begin{bmatrix}\underline{\tilde{\boldsymbol{u}}}^{\,0} \\ \underline{\boldsymbol{u}}^{\,\mathrm{gl}}\end{bmatrix}
\end{gathered}}$$

Computation of $\boldsymbol{u}_G$ involves forward- and back-substitutions on $\boldsymbol{K}^0$

# Non-Intrusive Implementation in Existing FEM Codes

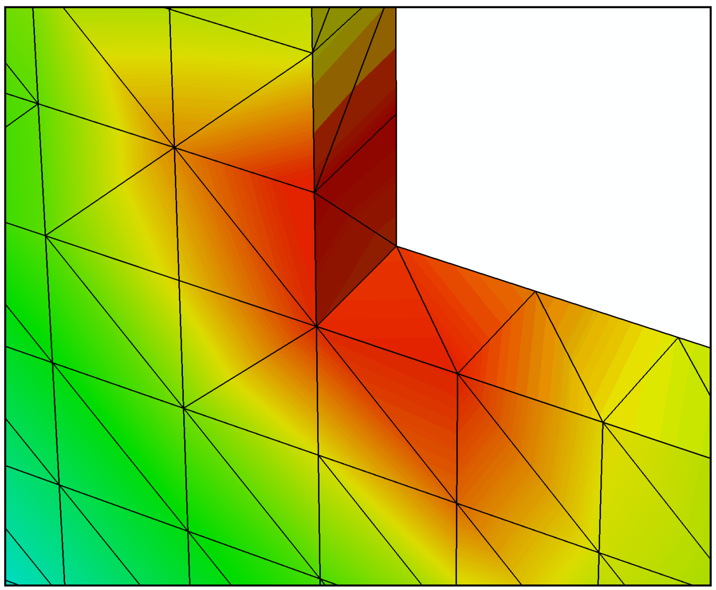
- Different solvers can be used to solve coarse and fine-scale problems



# Non-intrusive implementation of GFEM$^{gl}$ for Poisson equation in Abaqus *

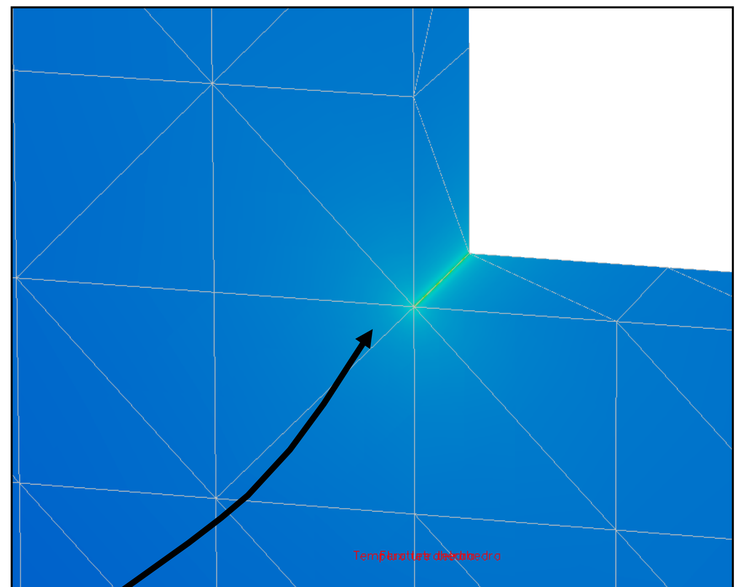
- Singularities in thermal fields

Coarse-Scale Abaqus Solution
at critical region



$$e^{r}_{U,IG} = 11.9\%$$

Abaqus + GFEM Solution



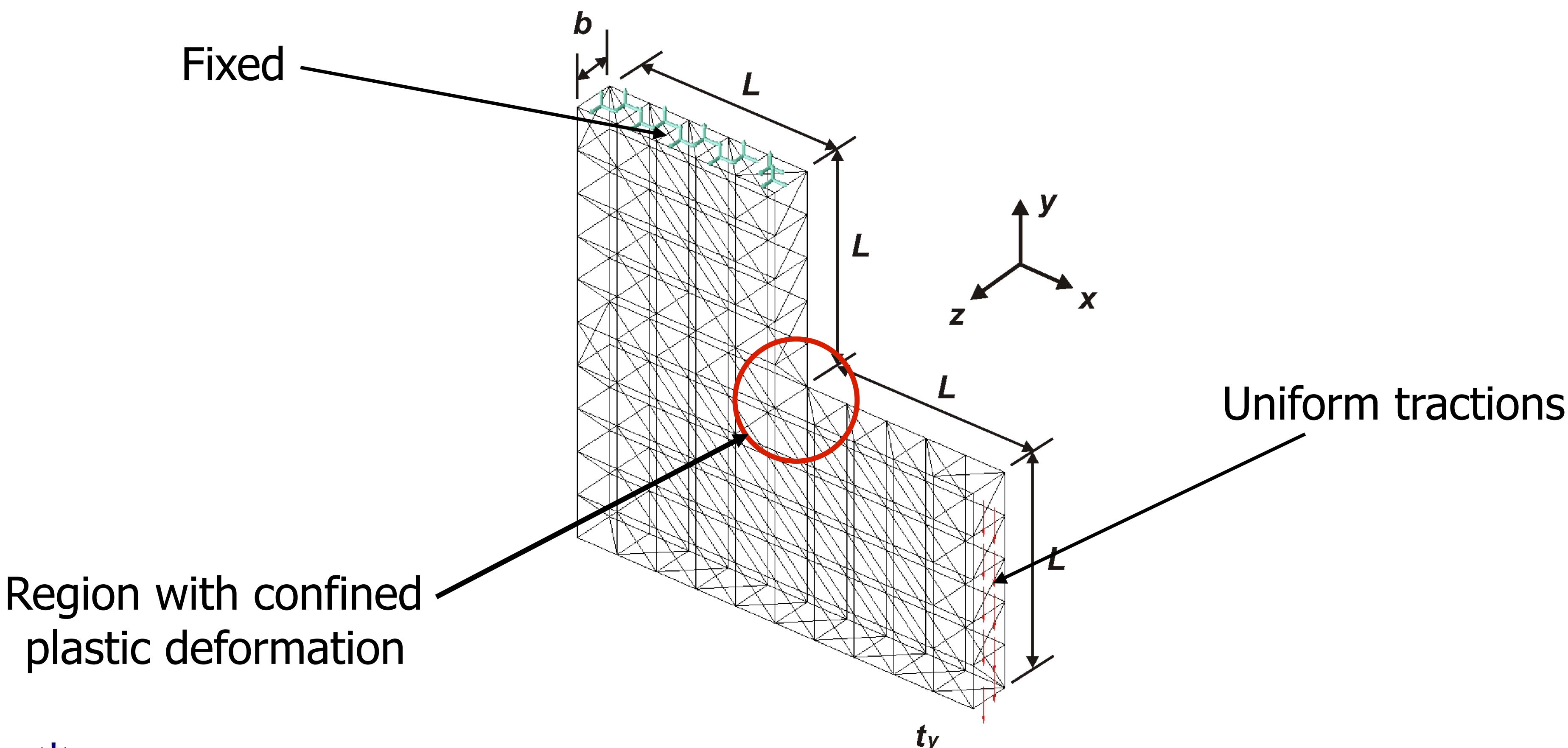
$$e^{r}_{U,EG} = 0.63\%$$

- Able to effectively capture sharp flux singularity adding only 16 global-local degrees of freedom to Abaqus model

*with J. Plews and T. Eason

# Enrichment Functions for Confined Plasticity Problems *

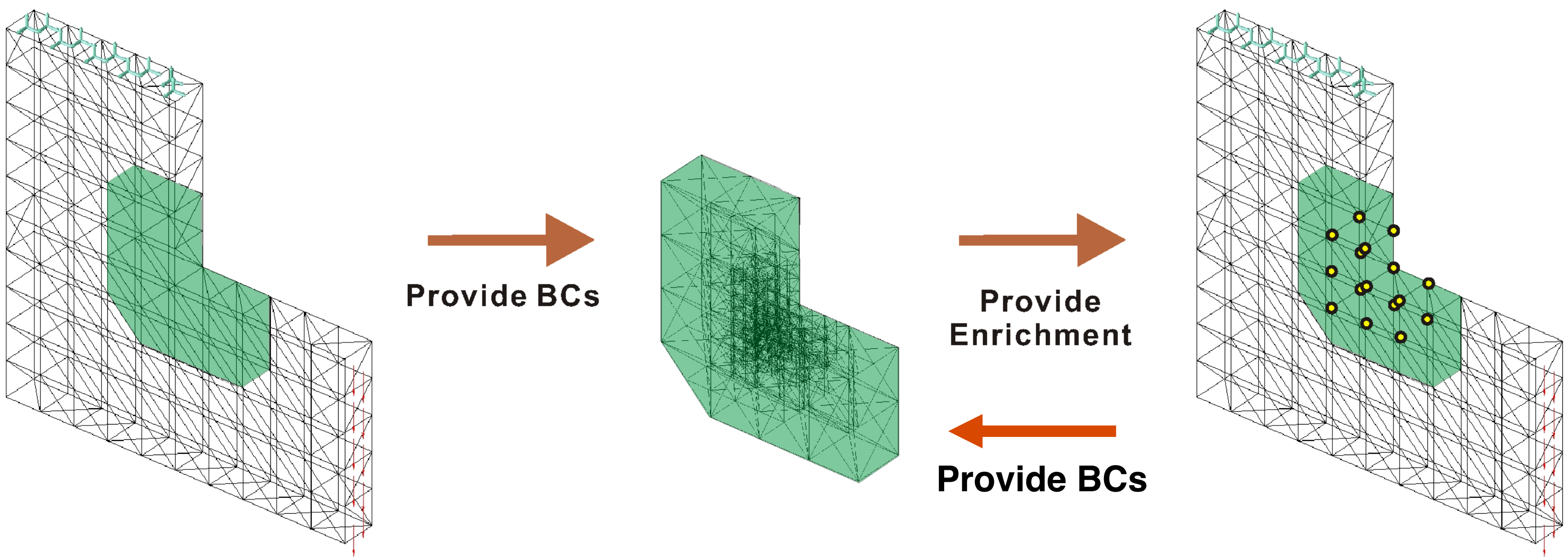
- **$J_2$ plasticity with isotropic hardening**




*with D.-J. Kim and S. Proenca

# Enrichment Functions for Confined Plasticity Problems

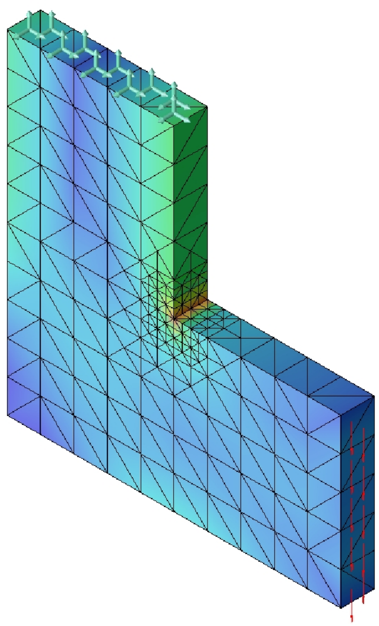
- **Key Idea:**
  - Use nonlinear local solution as enrichment for global problem solved on a *coarse mesh*



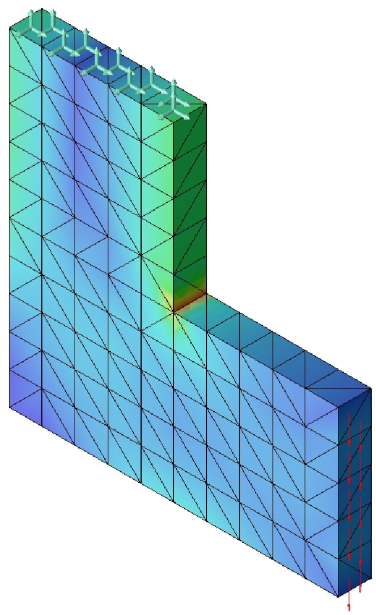

(a) Linear initial global problem (b) Nonlinear local problem (c) Nonlinear enriched global problem

  - G-L enrichments can be updated during iterative solution of nonlinear global problem

# Enrichment Functions for Confined Plasticity Problems

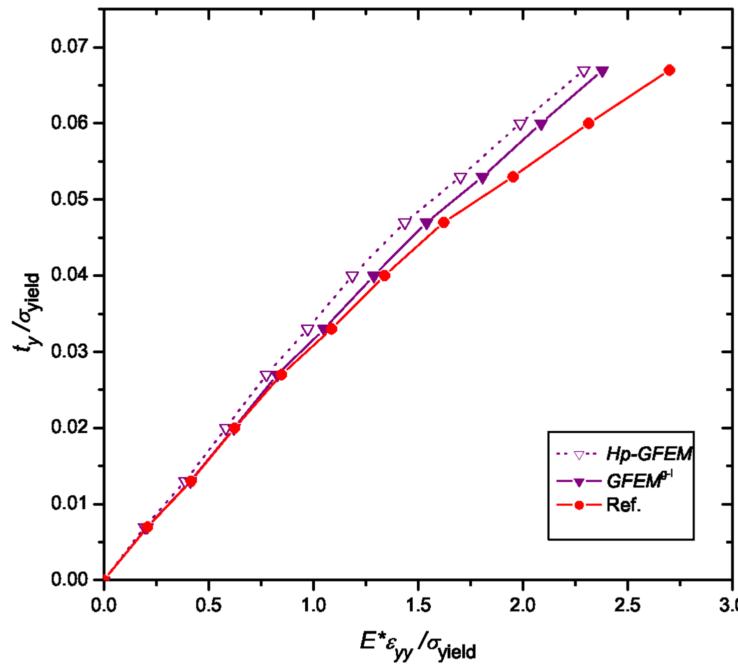


(a) *Hp-GFEM*



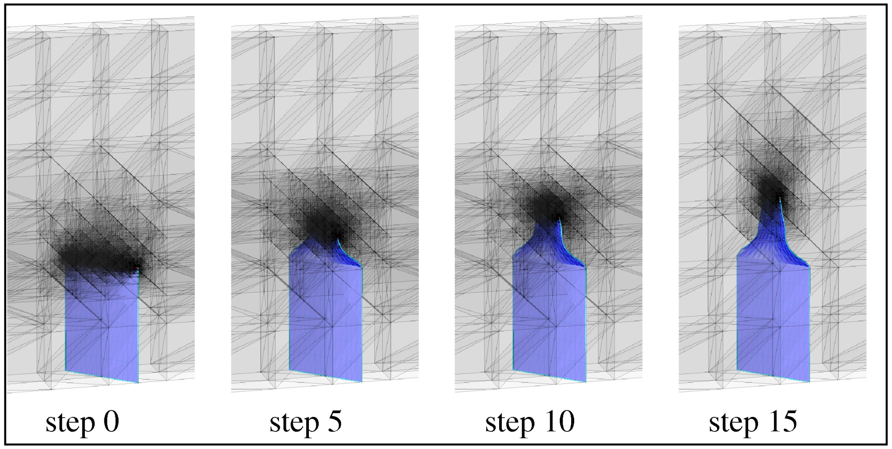
(b) *GFEM*$^{g\text{-}l}$

Von Mises stress distribution at final load step

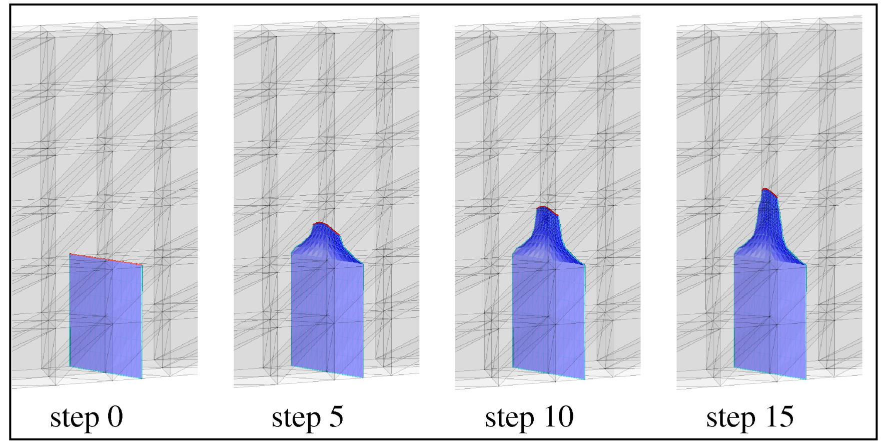



Load-displacement curves for *hp*-GFEM and GFEM$^{gl}$

# Concluding Remarks

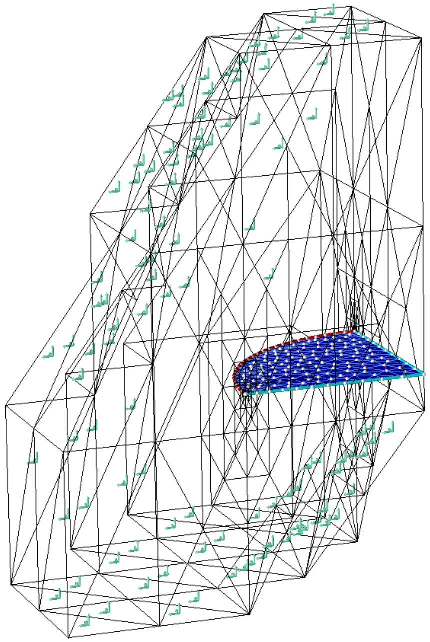



Available methods require AMR




Multiscale Generalized FEM

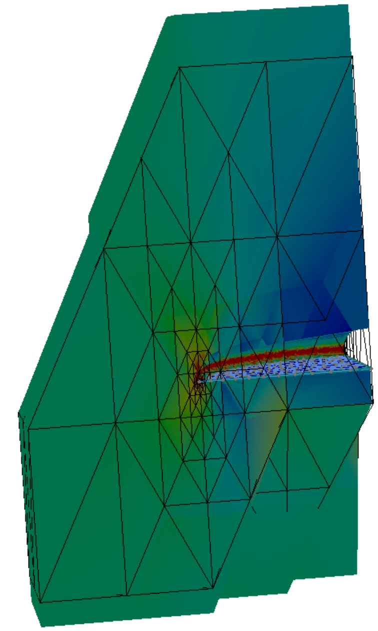
- FAST: Coarse-scale model of much reduced dimension than FEM; Fine-Scale computations are intrinsically parallelizable; recycle coarse scale solution
- ACCURATE: Can deliver same accuracy as adaptive mesh refinement (AMR) on meshes with elements that are orders of magnitude larger than in the FEM
- STABLE: Uses single-field variational principles
- TRANSITION: Fully compatible with FEM



# *Questions?*

caduarte@uiuc.edu

http://netfiles.uiuc.edu/caduarte/www/



**Support:**

AF OFFICE OF SCIENTIFIC RESEARCH

AIR FORCE RESEARCH LABORATORY