Analysis of Three-Dimensional Propagating Cracks: A Two-Scale Approach Using Coarse Finite Element Meshes

C. Armando Duarte

Jeronymo Pereira and Dae-Jin Kim

Dept. Civil and Environmental Engineering
University of Illinois at Urbana-Champaign, Urbana, IL

ECCM 2010 – Mini-Symposium GFEM and Numerical Treatment of Singularities
May 16-21, 2010 – Paris, France

Outline

- Generalized finite element methods: Basic ideas
- Global-local enrichments for 3-D Crack Growth
- Applications
- Assessment and closing remarks

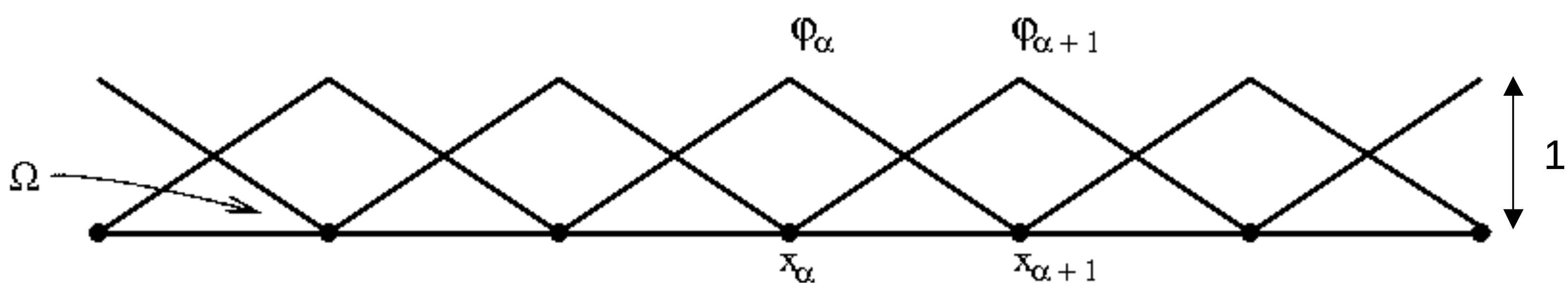
Generalized Finite Element Method

GFEM can be interpreted as a FEM with shape functions built using the concept of a partition of unity

Partition of Unity (PoU)

$$\boxed{\sum_{\alpha} \varphi_{\alpha}(x) = 1 \qquad \forall \; x \in \Omega}$$

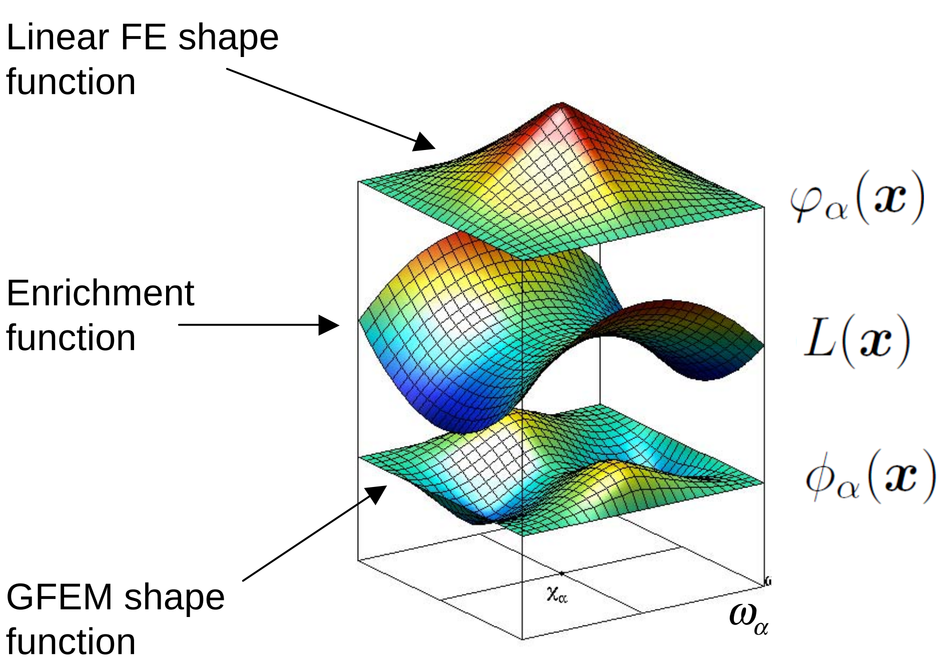
- φ_{α} = Linear FEM shape function



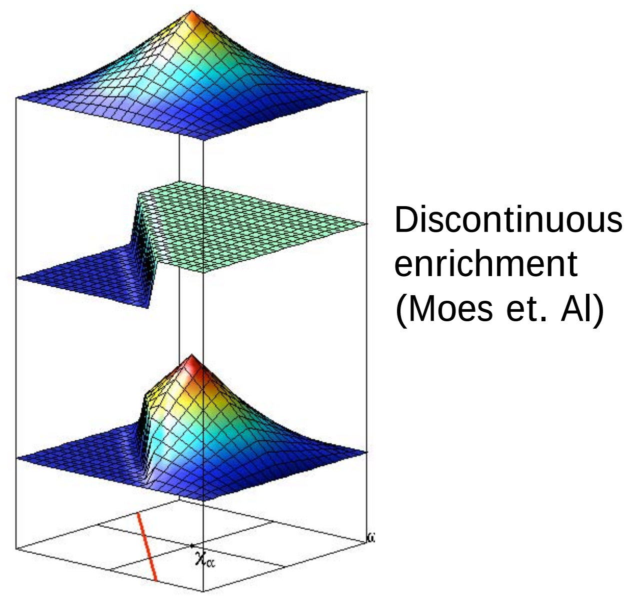
Generalized Finite Element Method

- GFEM shape function = FE shape function * enrichment function

$$\boxed{\phi_\alpha(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x}) L(\boldsymbol{x})}$$

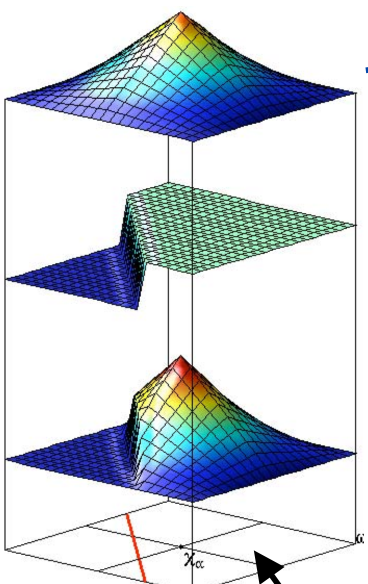


• Allows construction of shape functions which represent well the physics of the problem

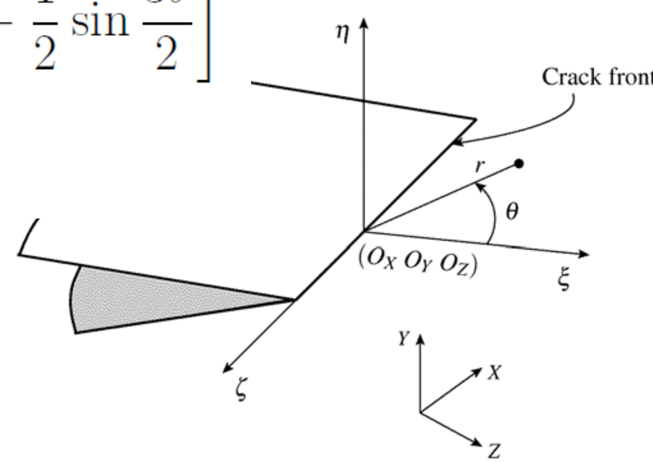


hp-GFEM Solution Space for 3-D Cracks

$$\boldsymbol{X}^{hp}(\Omega) = \left\{ \boldsymbol{u} = \sum_{\alpha=1}^{N} \underbrace{\varphi_\alpha(\boldsymbol{x})}_{\text{PoU}} \left[\underbrace{\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{polynomial}} + \underbrace{\mathcal{H}\tilde{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{discontinuous}} + \underbrace{\breve{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{singular}} \right] \right\}$$



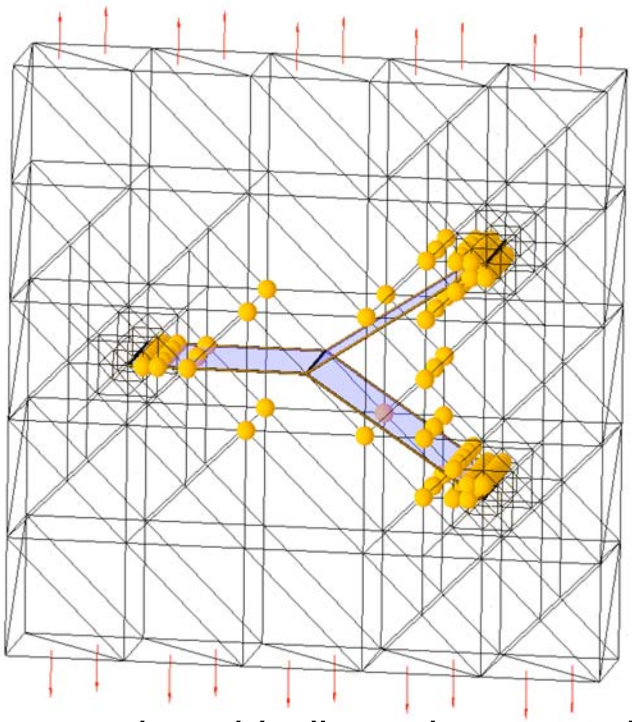
$$\begin{aligned}
\breve{L}^{\xi}_{\alpha 1}(r,\theta) &= \sqrt{r}\left[(\kappa - \frac{1}{2})\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right] \\
\breve{L}^{\eta}_{\alpha 1}(r,\theta) &= \sqrt{r}\left[(\kappa + \frac{1}{2})\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2}\right] \\
\breve{L}^{\zeta}_{\alpha 1}(r,\theta) &= \sqrt{r}\sin\frac{\theta}{2}
\end{aligned}$$



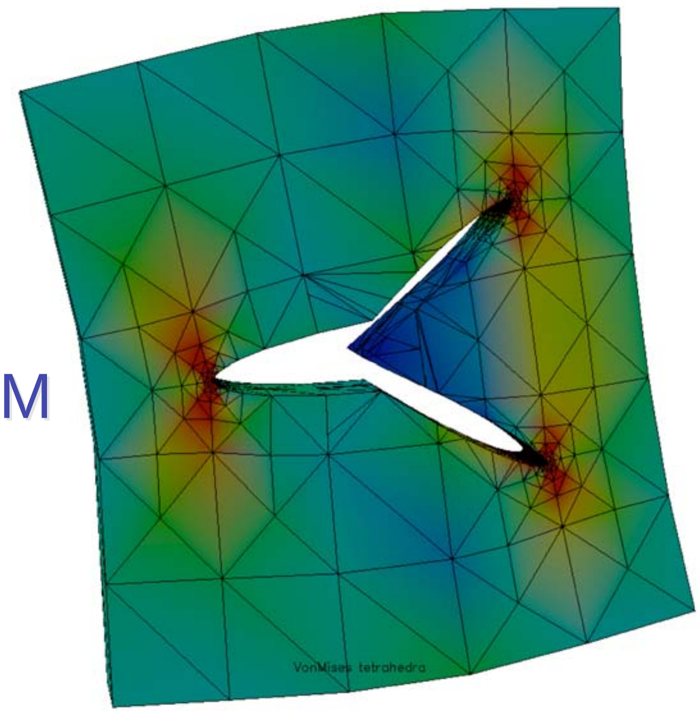
cloud or patch $'\alpha$

Modeling Cracks with hp-GFEM

- Discontinuities modeled via enrichment functions, *not* the FEM mesh
- Elements faces need not fit crack surfaces as in std FEM:
  Elements with good aspect ratio



hp-GFEM



● = Nodes with discontinuous enrichments

Von Mises stress

[Duarte et al., International Journal Numerical Methods in Engineering, 2007]

Application to Crack Fatigue Crack Growth

- **Edge-Notched Beam with Slanted:** *hp*-GFEM solution

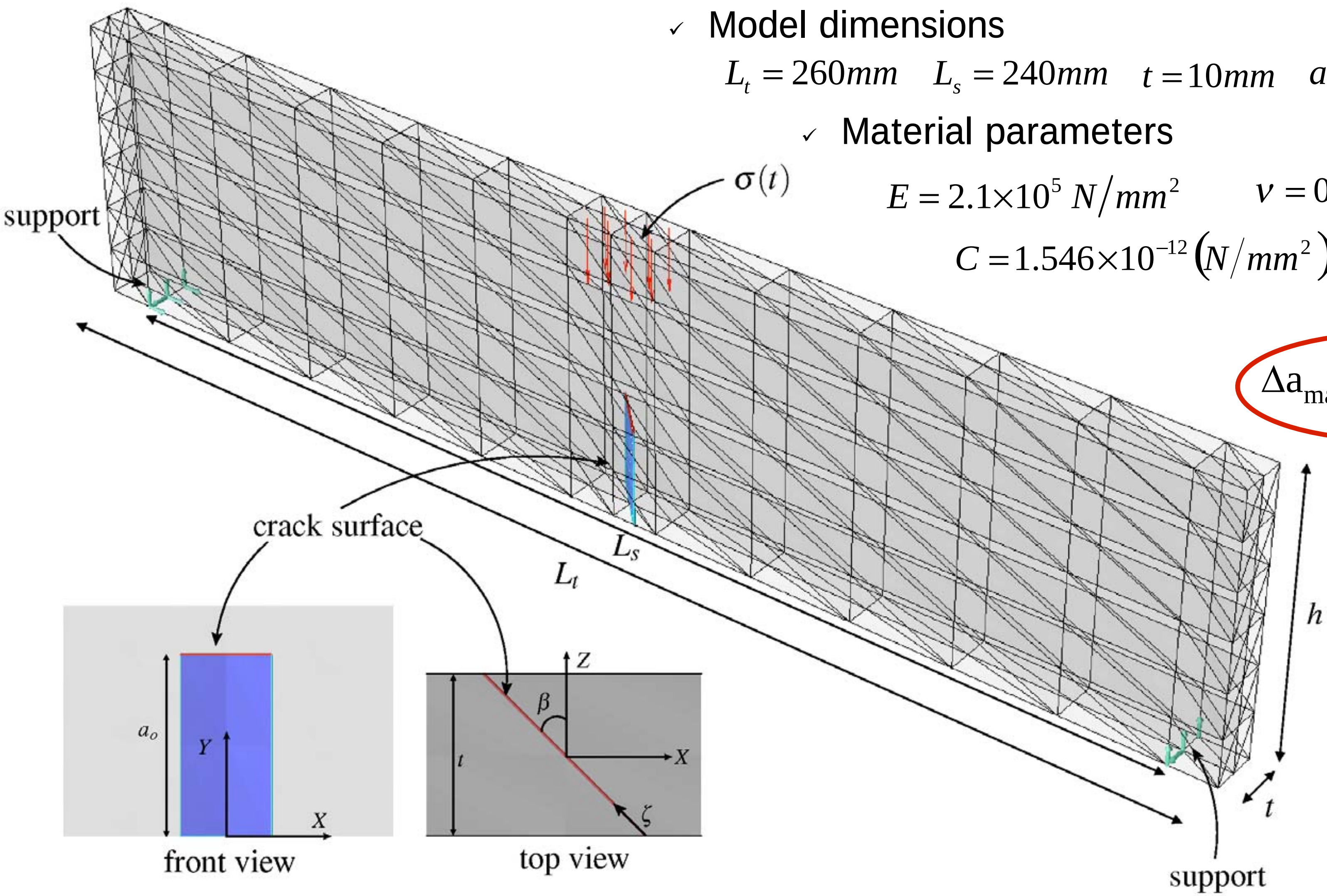
✓ Model dimensions

$L_t = 260mm \quad L_s = 240mm \quad t = 10mm \quad a_o/h = 1/3 \quad \beta = 45^o$

✓ Material parameters

$E = 2.1\times10^5\ N/mm^2 \qquad \nu = 0.3 \qquad m = 2.1$

$C = 1.546\times10^{-12}\left(N/mm^2\right)^{-2.1} mm^{-0.05}/cycle$



$\Delta a_{max} = 0.048\ a_o$

Movie

Assessment

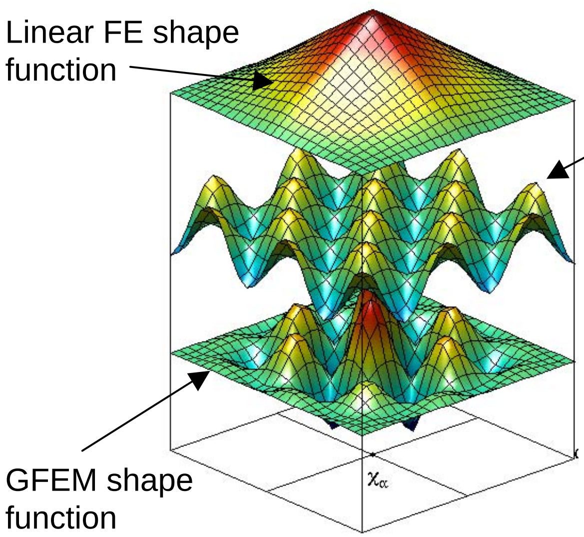
- Greatly facilitates discretization of cracks:
  - ✓ Simply insert crack surface in un-cracked mesh
  - ✓ Mesh need not fit crack surface: More robust than FEM
- Computational cost still high
  - ✓ Requires refinement of global mesh for each crack configuration
  - ✓ Needs to solve, large, global problem from scratch
- How to overcome these limitations?

- Crack growth algorithms require small crack increments, which lead to small changes in overall solution

- Take advantage of this: *Use available information to build solution space for next crack step*

Global-Local Enrichment Functions

- Enrichment functions computed from solution of local boundary value problems: Global-Local enrichment functions



Enrichment = Numerical solutions of BVP

[Copps et al. 2000],

[Duarte et al. 2005]

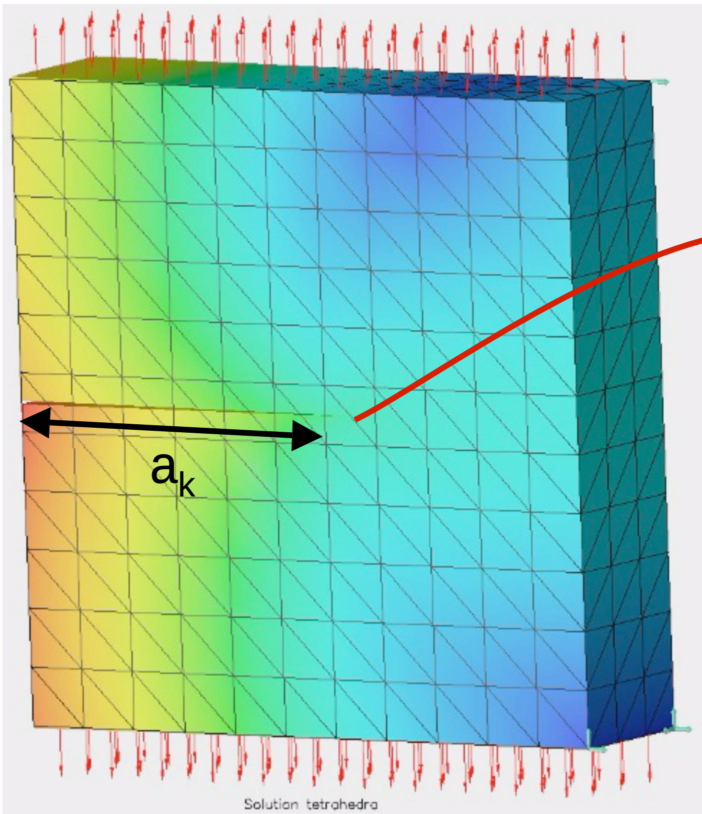
Instead of using analytically defined functions:

- Enrichment functions are produced numerically on-the-fly through a global-local analysis
- Use a *coarse* mesh enriched with Global-Local (G-L) functions

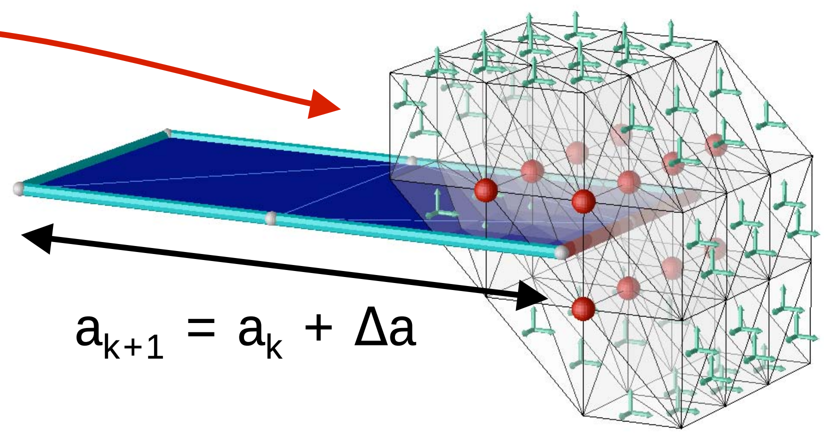
- Duarte and Kim, *Computer Methods in Applied Mechanics and Engineering*, 2008.
- O'Hara, Duarte and Eason, *Computer Methods in Applied Mechanics and Engineering*, 2009.

Global-Local Enrichments for 3-D Fractures

- u_G^k **solution of** global problem at crack step k



- Define local domain containing crack front

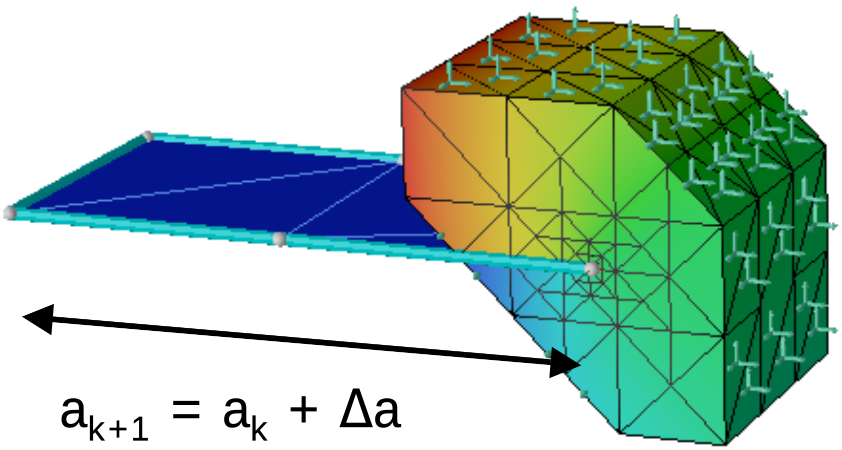


Local problem with crack size a_{k+1}

$u_G^k \in X_G^k(\Omega)$ = solution of global problem with crack size a_k

Global-Local Enrichments for 3-D Fractures

- Solve local problem at step k using *hp*-GFEM



$a_{k+1} = a_k + \Delta a$

Boundary conditions for local problems provided by global solution:

$$\boldsymbol{u}_L^k = \boldsymbol{u}_G^k \quad \text{on} \quad \partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)$$

$\boldsymbol{X}_L^k \left(\Omega_L^k\right)$ = *hp*-GFEM space

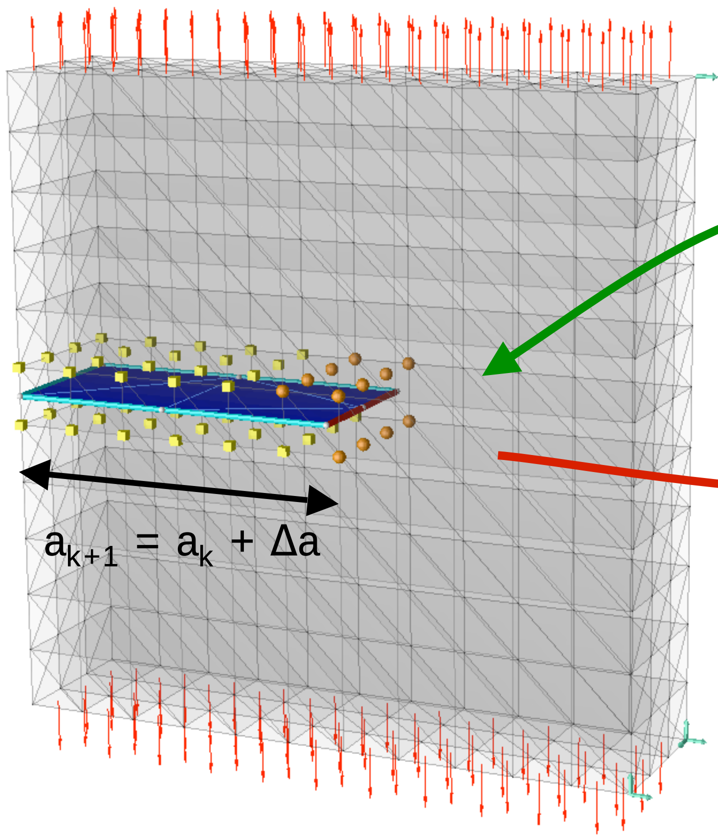
Find $\boldsymbol{u}_L^k \in \boldsymbol{X}_L^k \left(\Omega_L^k\right) \subset H^1 \left(\Omega_L^k\right)$ such that $\forall\, \boldsymbol{v}_L^k \in \boldsymbol{X}_L^k \left(\Omega_L^k\right)$

$$\int_{\Omega_L^k} \boldsymbol{\sigma}(\boldsymbol{u}_L^k) : \boldsymbol{\varepsilon}(\boldsymbol{v}_L^k) d\boldsymbol{x} + \kappa \int_{\partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)} \boldsymbol{u}_L^k \cdot \boldsymbol{v}_L^k ds$$

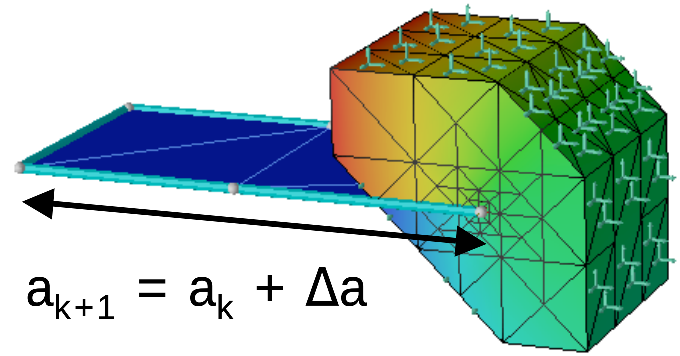
$$= \int_{\partial\Omega_L^k \cap \partial\Omega^\sigma} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}_L^k ds + \kappa \int_{\partial\Omega_L^k \setminus (\partial\Omega_L^k \cap \partial\Omega)} \boldsymbol{u}_G^k \cdot \boldsymbol{v}_L^k ds$$

Global-Local Enrichments for 3-D Fractures

- **Defining Step**: Global space is enriched with local solutions

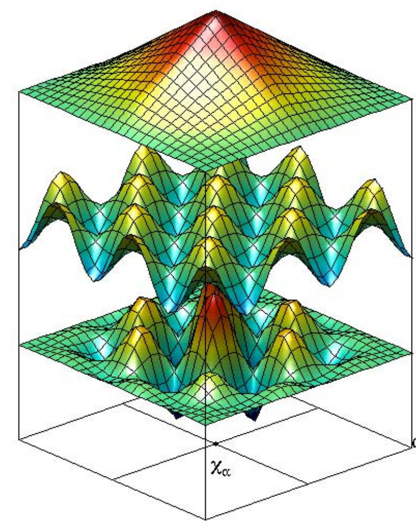


global-local
enrichments



BC for local
problem

$$\boxed{\phi_\alpha^{k+1}(\boldsymbol{x}) = \varphi_\alpha(\boldsymbol{x})\boldsymbol{u}_L^k(\boldsymbol{x})}$$

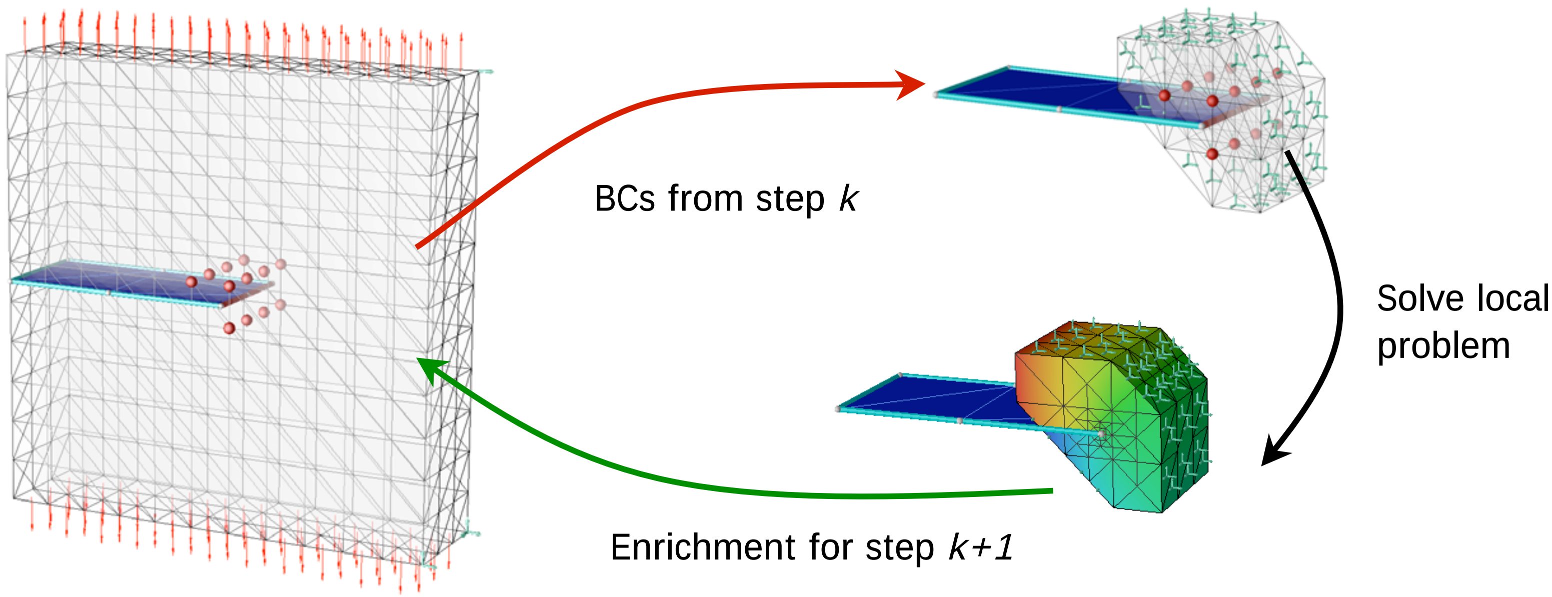


$\boldsymbol{u}_G^{k+1} \in \boldsymbol{X}_G^{k+1}(\Omega)$ = solution of global problem with crack size a$_{k+1}$

- Procedure may be repeated: Update local BCs and enrichment functions

Global-Local Enrichments for Crack Growth

- **Summary:** Use solution of global problem at crack step *k* to build enrichment functions for crack step *k+1*



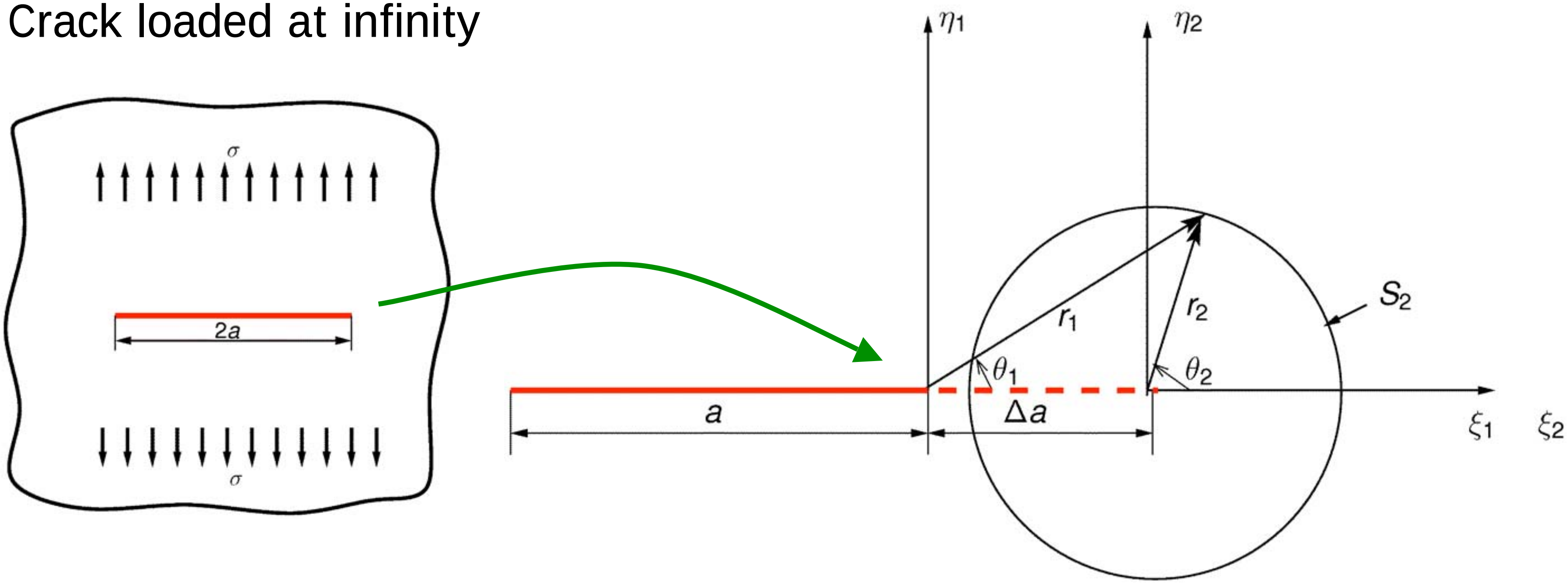
- Discretization spaces updated on-the-fly with global-local enrichment functions

$$\boldsymbol{X}_G^{k+1}(\Omega_G) = \left\{ \boldsymbol{u} = \underbrace{\sum_{\alpha=1}^{N} \varphi_\alpha(\boldsymbol{x})\hat{\boldsymbol{u}}_\alpha(\boldsymbol{x})}_{\text{coarse-scale approx.}} + \underbrace{\sum_{\beta \in \mathcal{I}_{gl}^k} \varphi_\beta(\boldsymbol{x})\boldsymbol{u}_\beta^{gl(k)}(\boldsymbol{x})}_{\text{fine-scale approx.}} \right\}$$

$\boldsymbol{u}_\beta^{gl(k)}$ = G-L enrichment

Error in Boundary Conditions

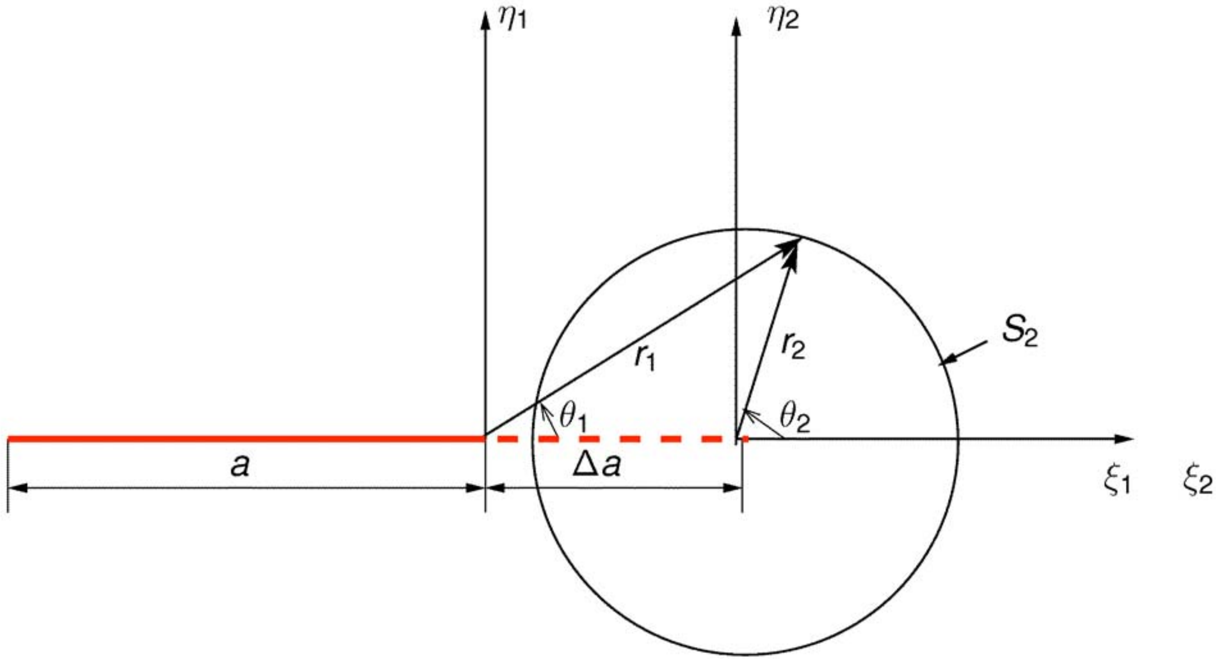
- Crack loaded at infinity



Solution for crack size $2a$

$$\boldsymbol{u}_a(r_1,\theta_1) = K_I^a\sqrt{r_1}\left\{\begin{array}{c} f_1^I(\theta_1) \\ f_2^I(\theta_1) \end{array}\right\}$$

Error in Boundary Conditions



Solution for crack size $2a + 2\Delta a$

$$\boldsymbol{u}_{a+\Delta a}(r_2, \theta_2) = K_I^{a+\Delta a} \sqrt{r_2} \begin{Bmatrix} f_1^I(\theta_2) \\ f_2^I(\theta_2) \end{Bmatrix}$$

Error in Boundary Conditions

Change in solution on S_2

$$\boldsymbol{e}(r_2, \theta_2) = \boldsymbol{u}_{a+\Delta a}(r_2, \theta_2) - \hat{\boldsymbol{u}}_a(r_2, \theta_2)$$

where

$$\hat{\boldsymbol{u}}_a(r_2, \theta_2) = \boldsymbol{u}_a \circ \boldsymbol{T}(r_2, \theta_2)$$

$$\boldsymbol{T} : (r_2, \theta_2) \mapsto (r_1, \theta_1)$$

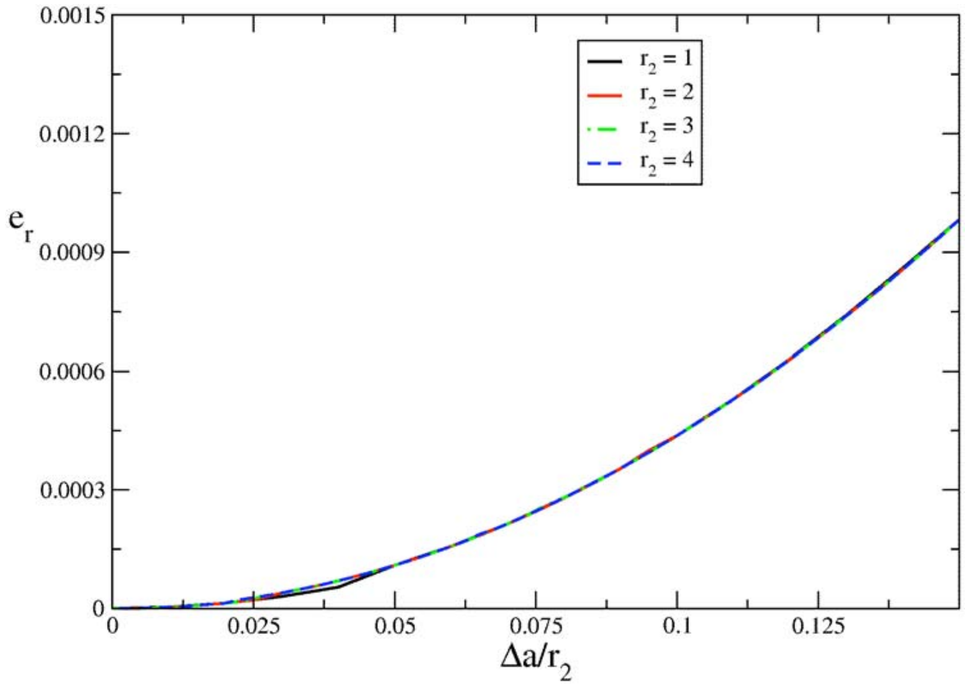
Let

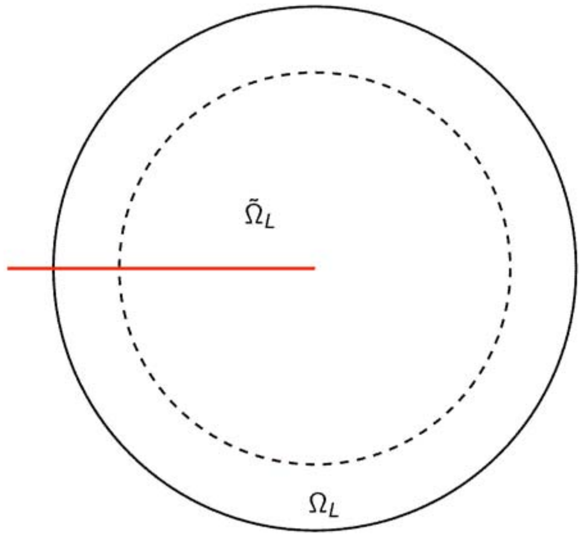
$$e_r = \frac{\|\boldsymbol{e}\|_{L^2(S_2)}}{\|\boldsymbol{u}_{a+\Delta a}\|_{L^2(S_2)}}$$

where

$$\|\boldsymbol{e}\|_{L^2(S_2)} = \sqrt{\int_{-\pi}^{\pi} \boldsymbol{e} \cdot \boldsymbol{e} \, d\theta_2}$$

Error in Boundary Conditions

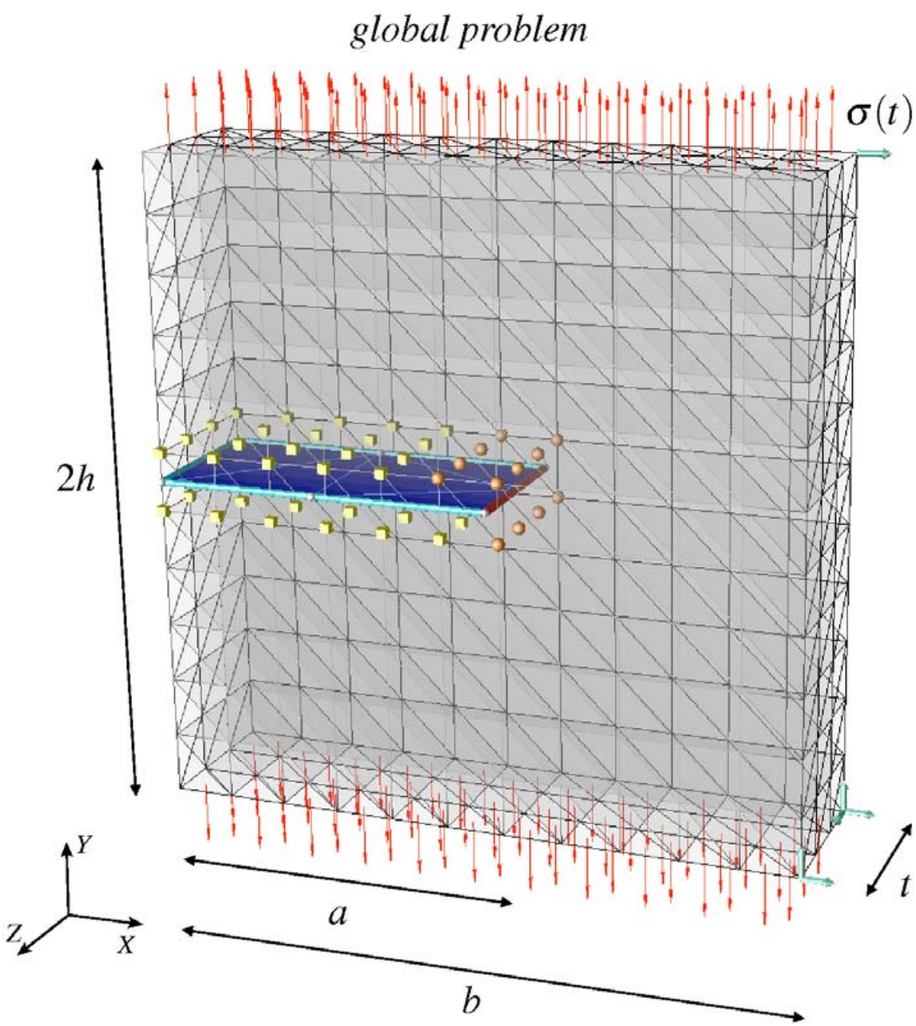




- Relative error scales with r_2 (distance of S_2 to crack tip): Error can be controlled using larger local domains
- Relative error is less than 10^{-4} for typical Δa
- Recall that error in boundary conditions can also be controlled through global-local-global cycles
- By Saint-Venant's principle (and homogeneous materials), the error of local problem solution due to errors in boundary conditions is small away from local boundary

GFEM^{gl} for crack growth - example

- **Panel with edge crack**



- Model dimensions

$$2h/t = b/t = 4 \qquad a/t = 2.1$$

- Material parameters

$$E = 1.0\times 10^{5}\, MPa \qquad \nu = 0.3$$

- Paris Law parameters (crack growth)

$$C = 1.5463\times 10^{-11}\, MPa^{-2.1} m^{-0.05} \qquad m = 2.1$$

$$\Delta a_{max} = 0.048\ a$$

- Reference solutions for strain energy and SIF
  - *hp*-GFEM with p=3 and plane-strain solution

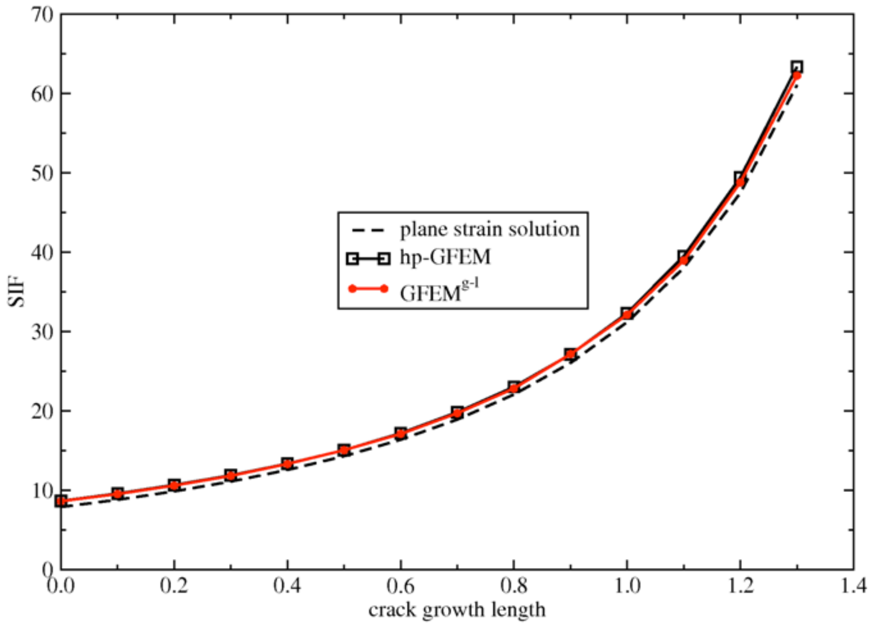
- Simulation output

local-problem

GFEM$^{g\text{-}l}$ vs. *hp*-GFEM

GFEM^{gl} vs. *hp*-GFEM

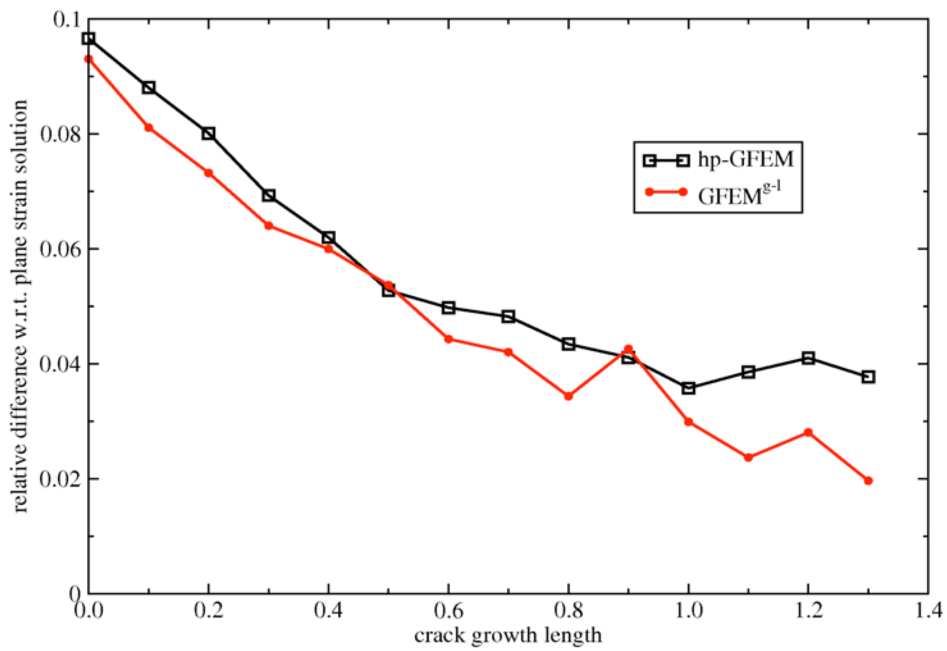
- Stress intensity factors at center of crack front



✓ Reduced number of *dofs*

- *hp*-GFEM: 35,157 *dofs* (average)
- GFEM^{gl}: 19,236 global *dofs* (average) <u>only 36 *dofs* from global-local</u>

- Relative difference w.r.t. pl strain



✓ Relative difference w.r.t. plane strain solution

✓ Both methods show good agreement

Edge-Notched Beam with Slanted Crack

- **Fatigue Crack Growth: GFEM^{gl} solution**

✓ Model dimensions

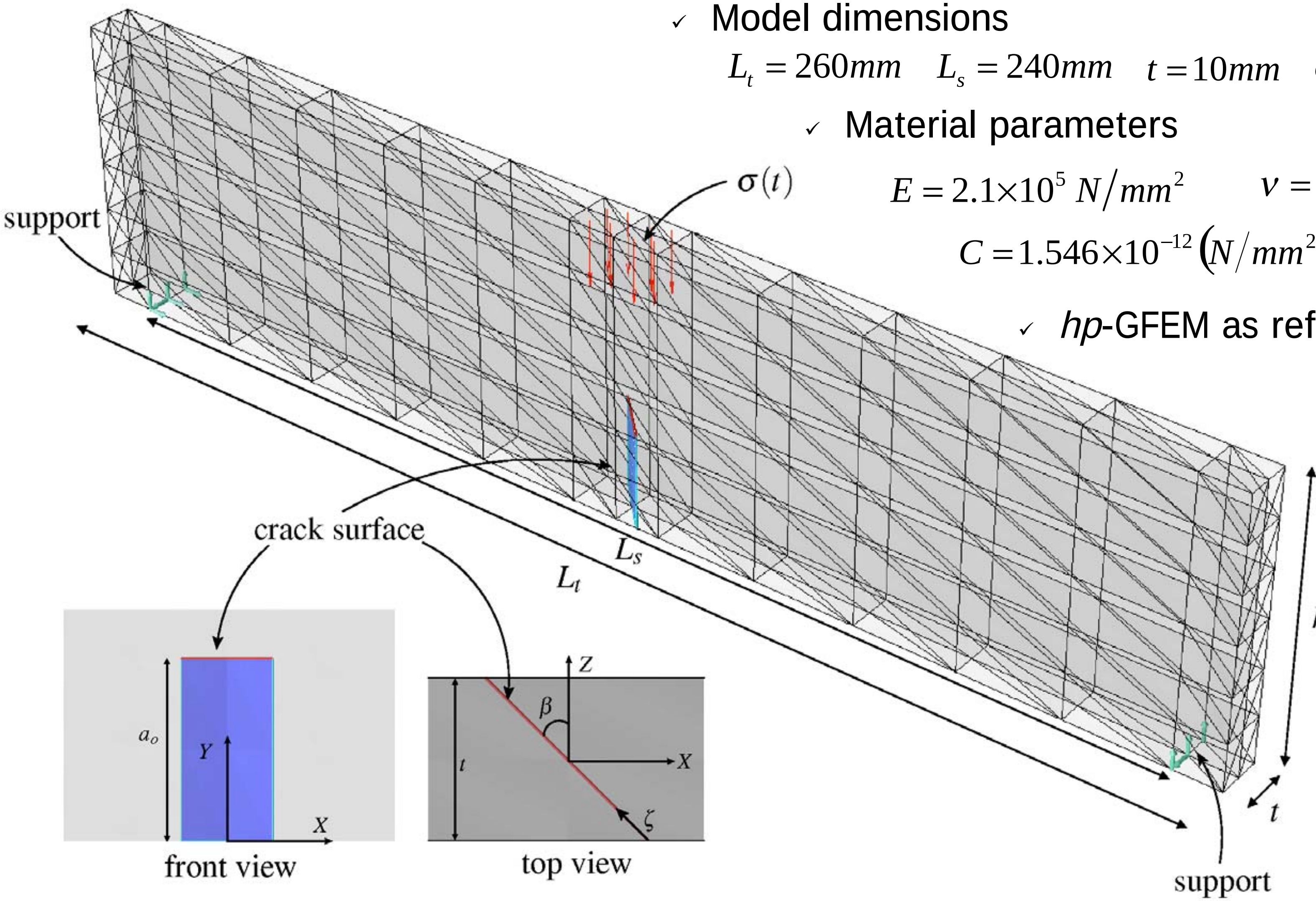
$$L_t = 260mm \quad L_s = 240mm \quad t = 10mm \quad a_o/h = 1/3 \quad \beta = 45^o$$

✓ Material parameters

$$E = 2.1\times10^5\ N/mm^2 \qquad \nu = 0.3 \qquad m = 2.1$$

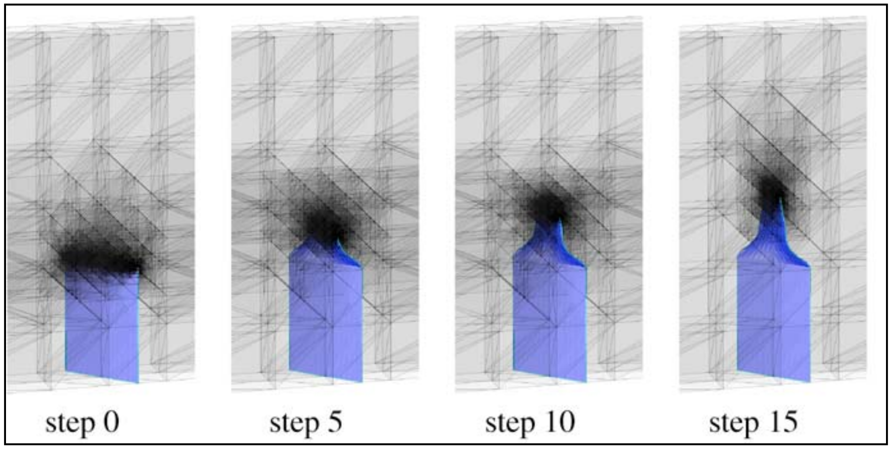
$$C = 1.546\times10^{-12}\left(N/mm^2\right)^{-2.1} mm^{-0.05}/cycle$$

✓ *hp*-GFEM as reference solution

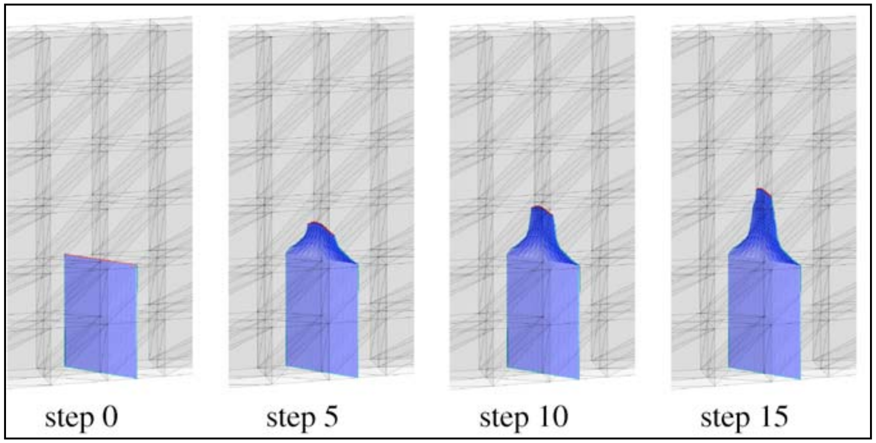


movie

Edge-Notched Beam with Slanted Crack



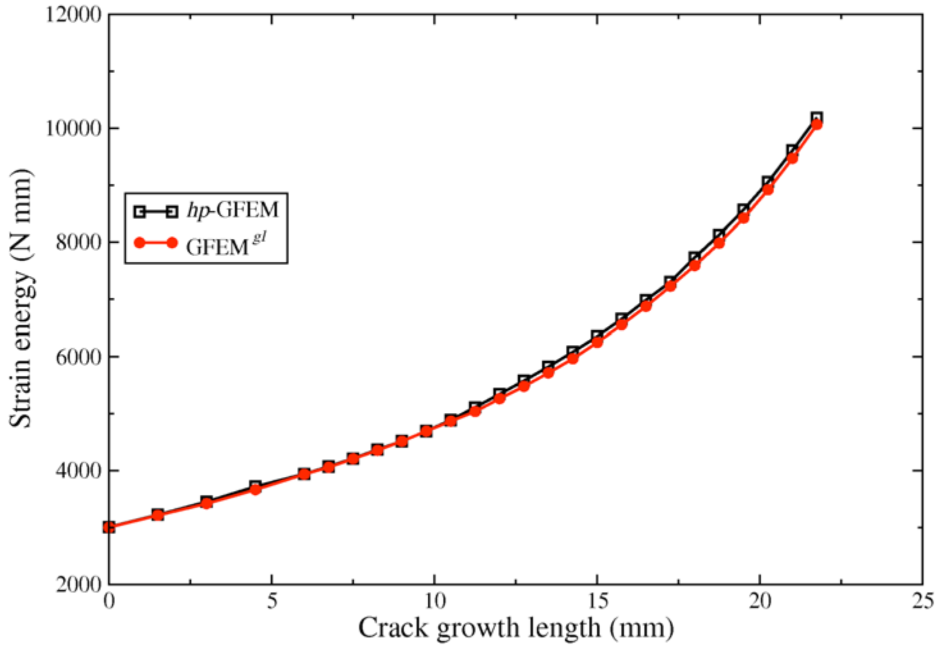
Available Methods – *hp*-GFEM/FEM



Two-Scale Generalized FEM – GFEM^{gl}

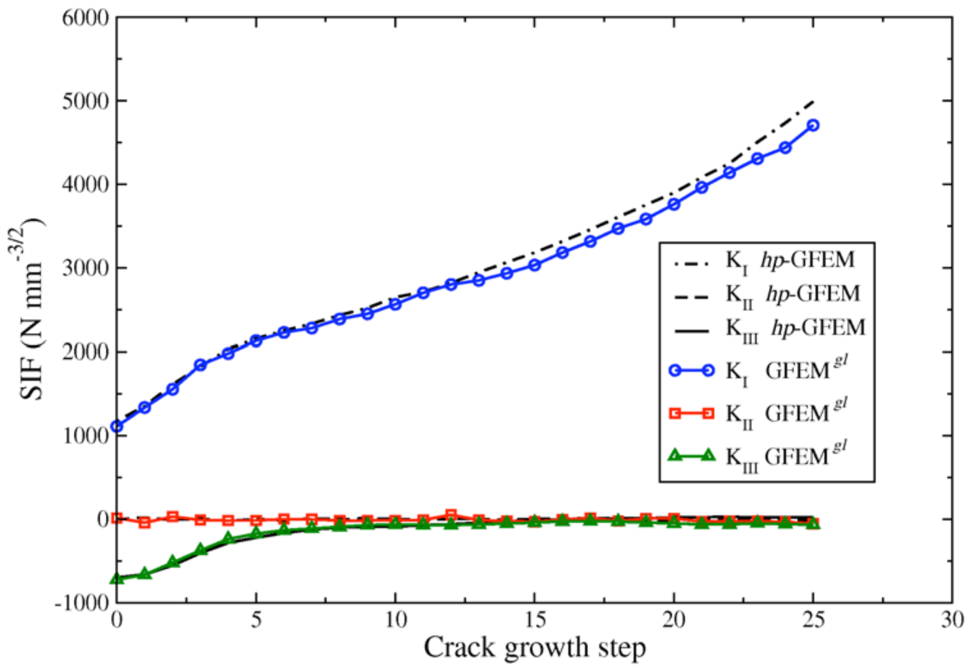
- Mesh with elements that are orders of magnitude larger than in a FEM mesh
- Fully compatible with FEM

Edge-Notched Beam with Slanted Crack



- **Strain energy**
  - Good agreement between GFEM^{gl} and *hp*-GFEM

- **Stress intensity factors**
  - SIFs at the middle of the crack front



Computation of Solution at a Crack Step

$$\boldsymbol{u}_G = \underbrace{\tilde{\boldsymbol{u}}^0}_{\text{coarse scale (polynomial)}} + \underbrace{\boldsymbol{u}^{\text{gl}}}_{\text{fine scale (G-L)}} = [\boldsymbol{N}^0 \boldsymbol{N}^{\text{gl}}] \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^{\,0} \\ \underline{\boldsymbol{u}}^{\,\text{gl}} \end{bmatrix}$$

$$\begin{aligned} \underline{\tilde{\boldsymbol{u}}}^{\,0} &= \text{DOFs associate with coarse scale discretization} \\ \underline{\boldsymbol{u}}^{\,\text{gl}} &= \text{DOFs associate with G-L (hierarchical) enrichments} \end{aligned}$$

$$\dim(\underline{\boldsymbol{u}}^{\,\text{gl}}) << \dim(\underline{\tilde{\boldsymbol{u}}}^{\,0})$$

This leads to

$$\begin{bmatrix} \boldsymbol{K}^0 & \boldsymbol{K}^{0,\text{gl}} \\ \boldsymbol{K}^{\text{gl},0} & \boldsymbol{K}^{\text{gl}} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^{\,0} \\ \underline{\boldsymbol{u}}^{\,\text{gl}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^0 \\ \boldsymbol{F}^{\text{gl}} \end{bmatrix}$$

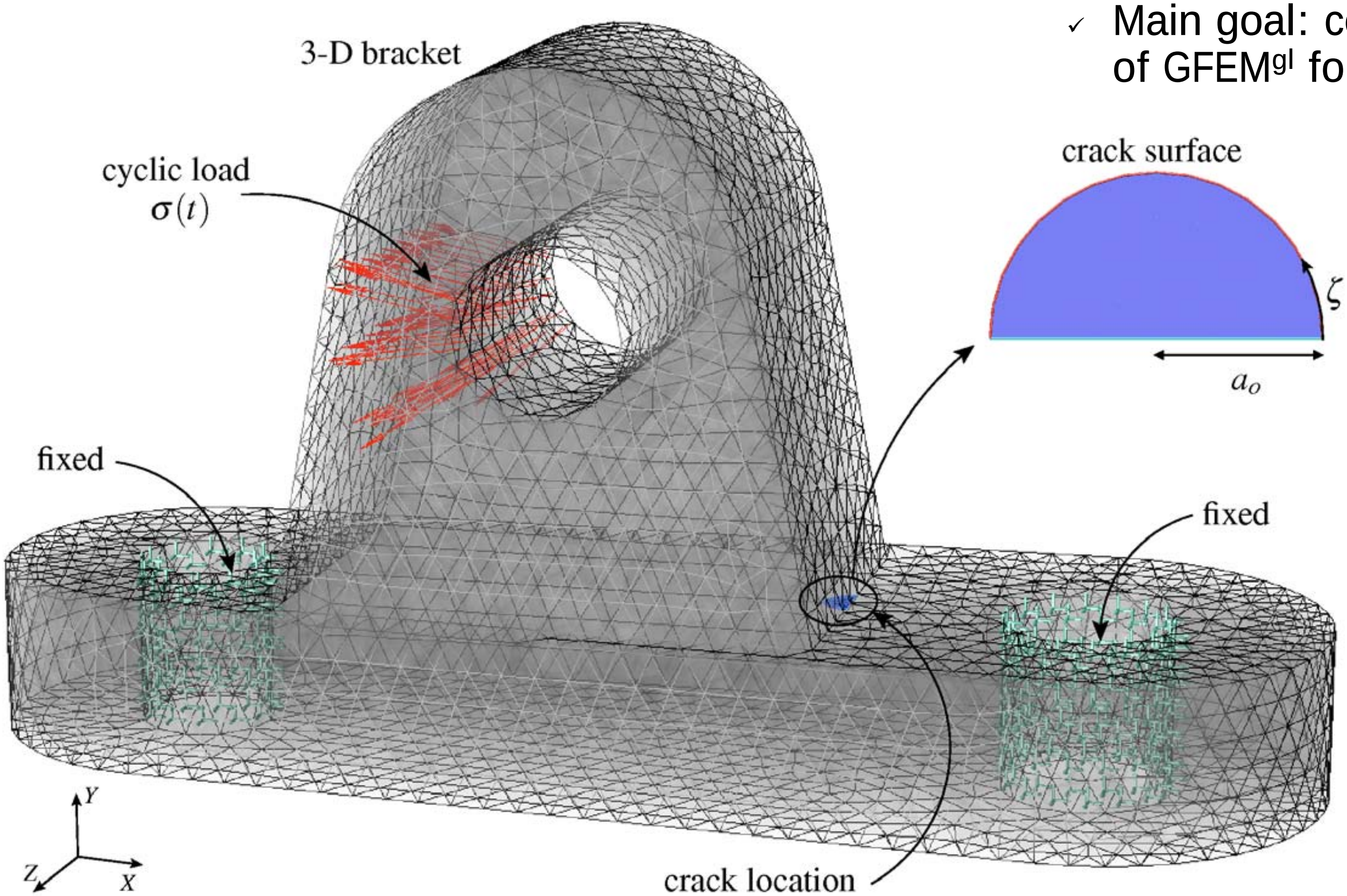
Solve using, e.g., static condensation of $\underline{\boldsymbol{u}}^{\,\text{gl}}$

Computational Efficiency

- **Bracket with half-penny shaped crack**

✓ *hp*-GFEM as reference solution

✓ Main goal: computational efficiency of GFEM^{gl} for crack growth

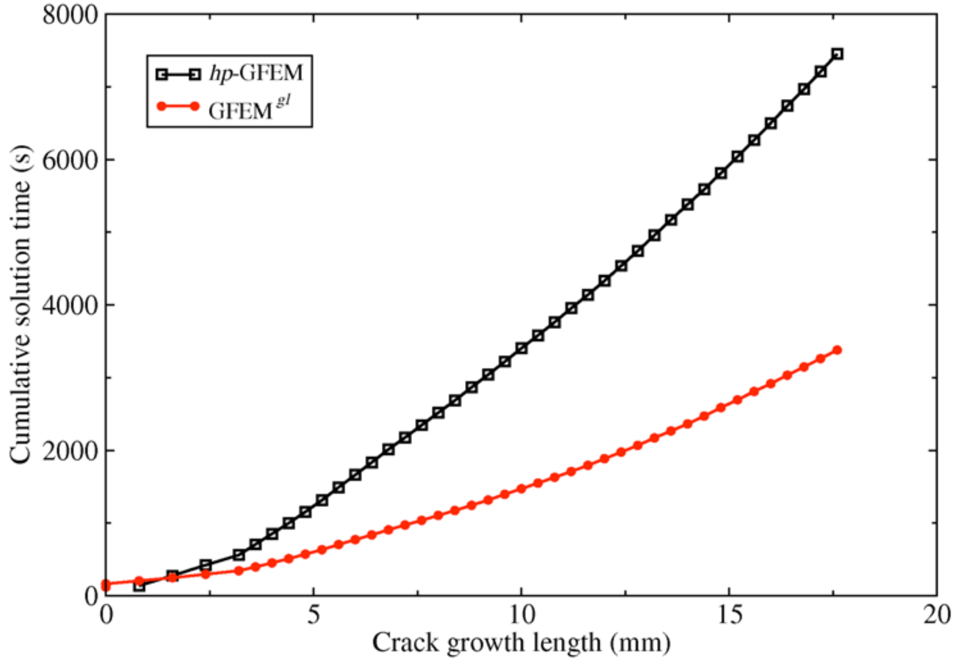


movie

3-D mesh courtesy of Altair Engineering

Computational Efficiency

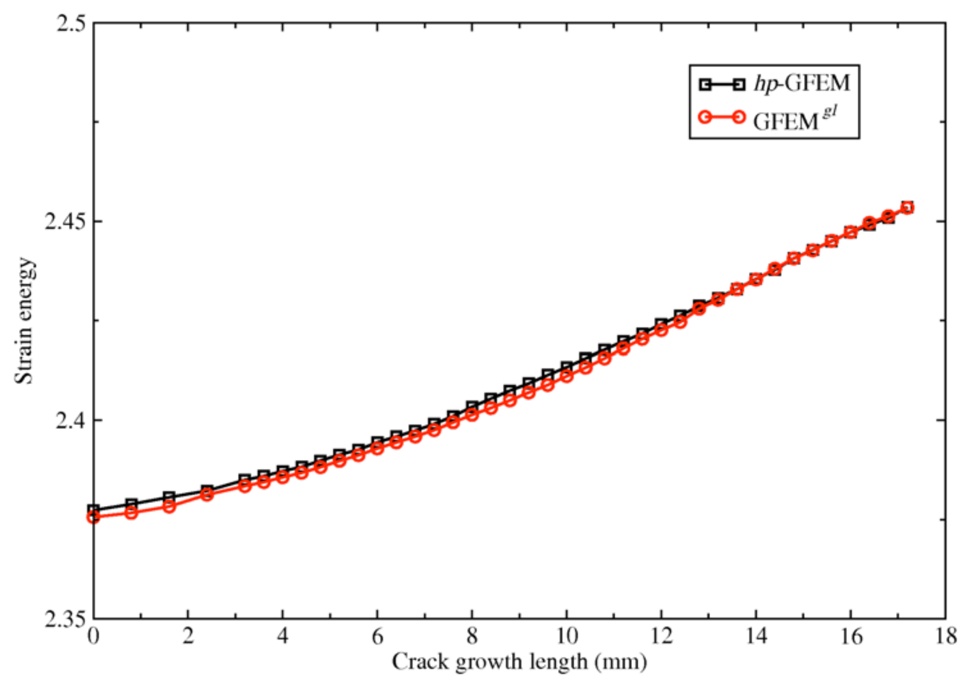
- **Computational cost analysis**



- ~ 60% computational cost reduction
- *hp*-GFEM and GFEM^{gl} solutions show good agreement

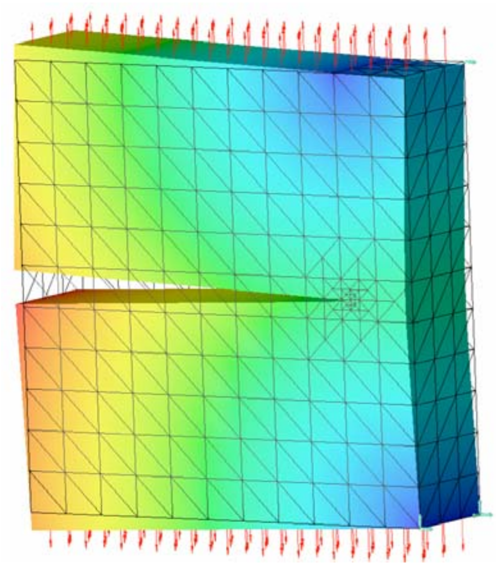
- GFEM^{gl}:
  115,470 + 27 *dofs* (min)
  115,470 + 84 *dofs* (max)
- *hp*-GFEM:
  186,666 global *dofs* (min)
  255,618 global *dofs* (max)

- **Strain Energy**

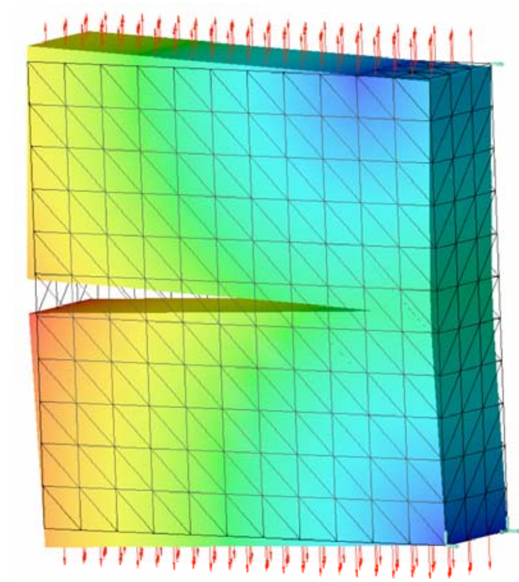


Concluding Remarks

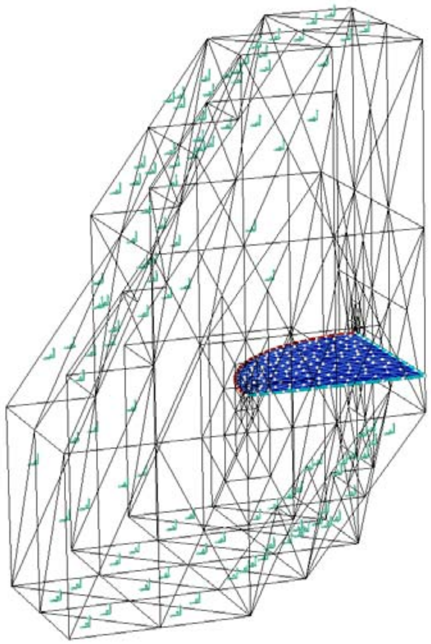
FEM with remeshing / *hp*-GFEM



GFEM with G-L enrichments

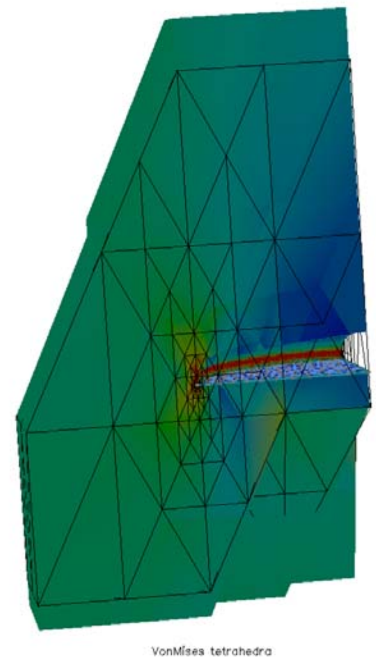


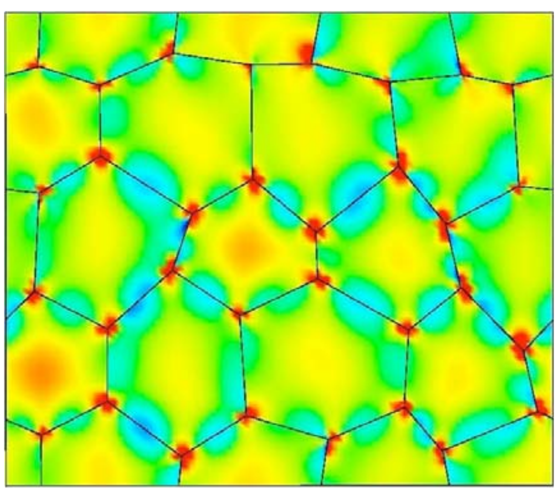
- The GFEM^{gl} is robust and accurate
  - remove FEM meshing issues in 3-D crack simulations
  - account for interaction among non-separable scales
- Computationally efficient
  - can deliver accurate solutions on coarse meshes
  - global matrices can be recycled during crack propagation simulations
- Can be applied to a broad range of problems: Fracture (linear and non-linear), time-dependent, etc.



Questions?

caduarte@illinois.edu





AF OFFICE OF SCIENTIFIC RESEARCH

AIR FORCE RESEARCH LABORATORY