Multiscale Analysis of Sharp Transient Thermal Gradients Using Coarse Generalized Finite Element Meshes

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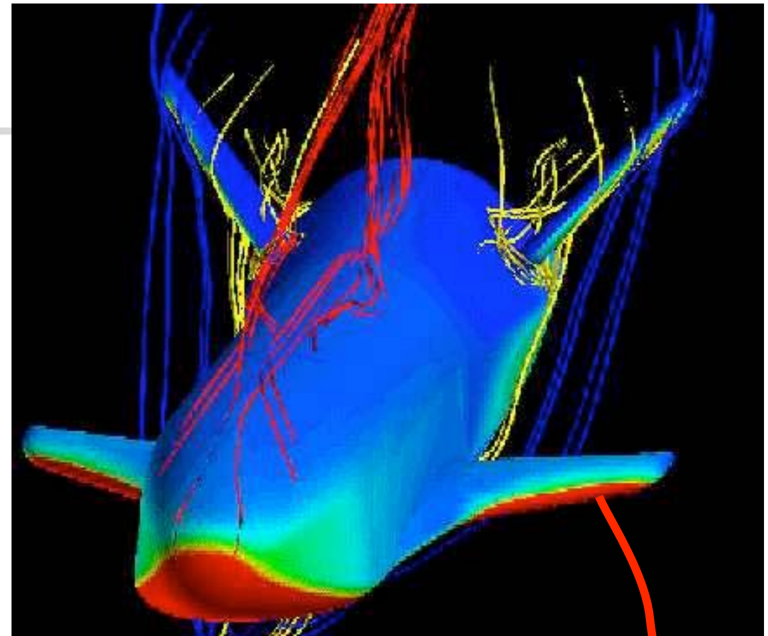
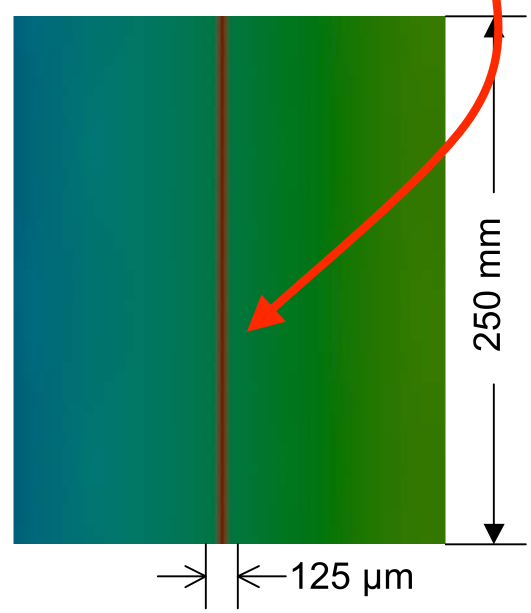
Motivation

- **Thermal loads on hypersonic aircrafts**



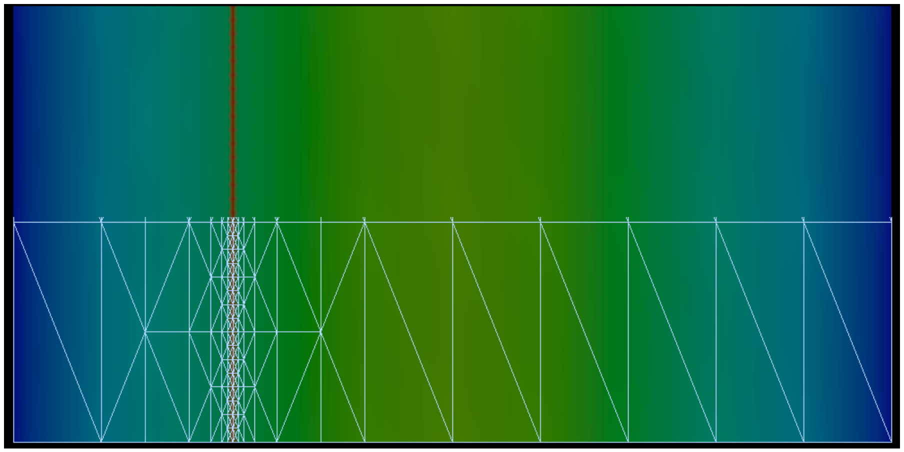
- Vehicles in hypersonic flight are subjected to intense thermal loads
- Shock wave impingements cause large thermal gradients
- Experiments are difficult and limited

- Predictive simulations require modeling phenomena spanning several spatial scales
- Advances in existing computational methods are needed
- Increasing computational power alone is not enough



Dimensions not to scale

Multi-Scale Problems: Computational Challenges



Thermal loads on hypersonic aircrafts

- Adaptive mesh refinement (AMR) is required in standard FE Methods
- AMR requires several solve/adapt cycles in large computational models
- Most fine scale effects are non-local
  - ➢ Standard global-local analysis is not robust in general
- Transient and/or non-linear effects add significant complexity
  - ➢ Fine meshes may require very small time steps for stability/accuracy

Objectives and Outline

- Capture multi-scale phenomena in structural scale FEM meshes
  - No refinement of large scale structural models
- Computational accuracy comparable to AMR but at a much reduced computational cost and complexity

Proposed approach

- Generalized FEM with global-local enrichment functions: GFEM^{gl}

Outline

- Generalized finite element methods: Basic ideas
- Global-local enrichments for sharp thermal gradients
- Applications
- Assessment and closing remarks

Early works on Generalized FEMs

- Babuska, Caloz and Osborn, 1994 (Special FEM).
- Duarte and Oden, 1995 (Hp Clouds).
- Babuska and Melenk, 1995 (PUFEM).
- Oden, Duarte and Zienkiewicz, 1996 (Hp Clouds/GFEM).
- Duarte, Babuska and Oden, 1998 (GFEM).
- Belytschko et al., 1999 (Extended FEM).
- Strouboulis, Babuska and Copps, 2000 (GFEM).

- Basic idea:
  - Use a partition of unity to build shape functions

- Recent review paper

  Belytschko T., Gracie R. and Ventura G. A review of extended/generalized finite element methods for material modeling, *Mod. Simul. Matl. Sci. Eng.*, 2009

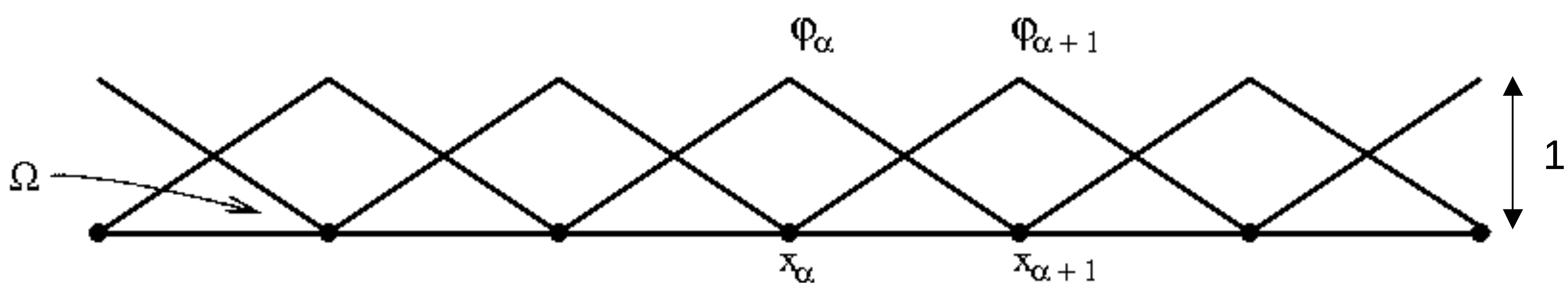
Generalized Finite Element Method

GFEM can be interpreted as a FEM in which shape functions are built using the concept of a partition of unity

Partition of Unity (PoU)

$$\boxed{\sum_{\alpha} \varphi_{\alpha}(x) = 1 \qquad \forall \; x \in \Omega}$$

- φ_{α} = Linear FEM shape function



Generalized Finite Element Method

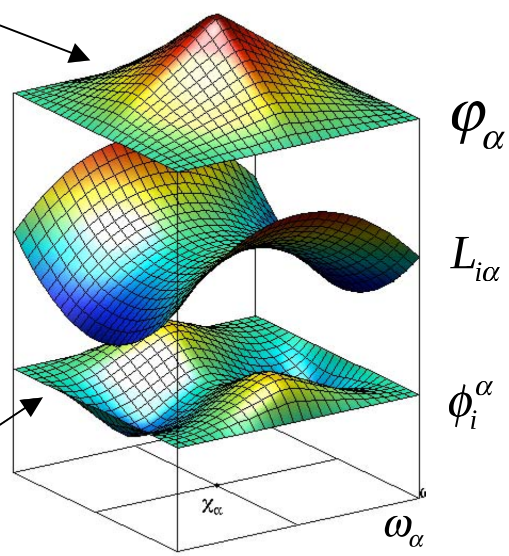
- GFEM shape function = FE shape function * enrichment function

$$\boxed{\phi_i^\alpha = \varphi_\alpha L_{i\alpha} \qquad i \in I(\alpha)}$$

Linear FE shape function

Enrichment function

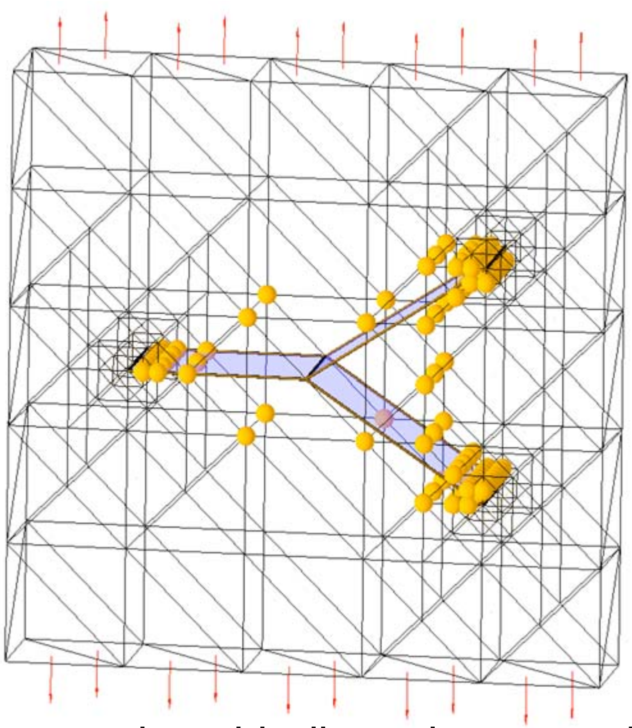
GFEM shape function



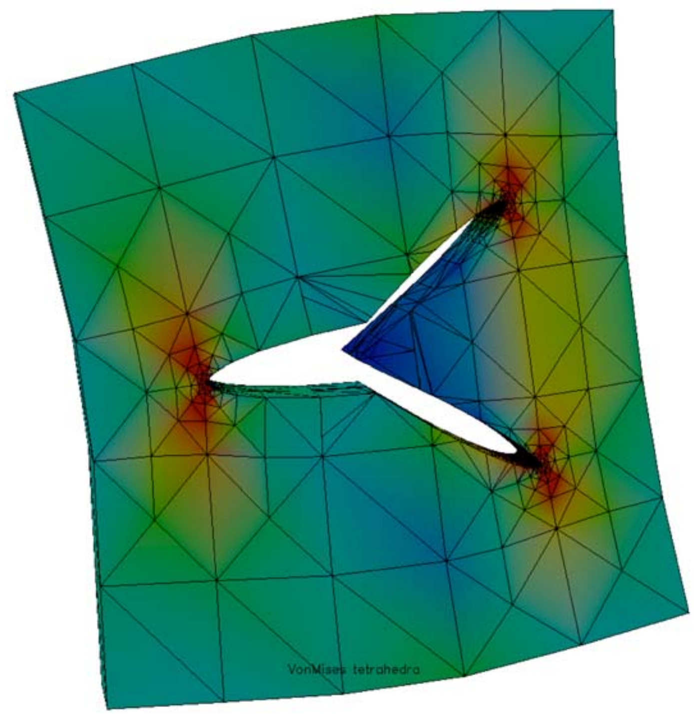
- Allows construction of shape functions which represent well the physics of the problem
- Applied to fracture mechanics, boundary and internal layers, moving interfaces, etc.

Modeling Cracks with the GFEM

- Discontinuities modeled via enrichment functions, *not* the FEM mesh
- Elements faces need not fit crack surfaces as in the std FEM:
  Elements with good aspect ratio



= Nodes with discontinuous enrichments

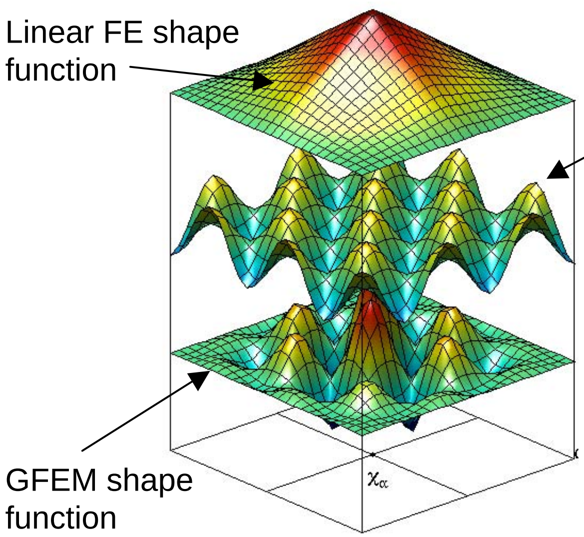


Von Mises stress

[Duarte et al., International Journal Numerical Methods in Engineering, 2007]

Global-Local Enrichment Functions

- Enrichment functions computed from solution of local boundary value problems: Global-local enrichment functions



Enrichment = Numerical solutions of BVP

[Copps et al. 2000],

[Duarte et al. 2005]

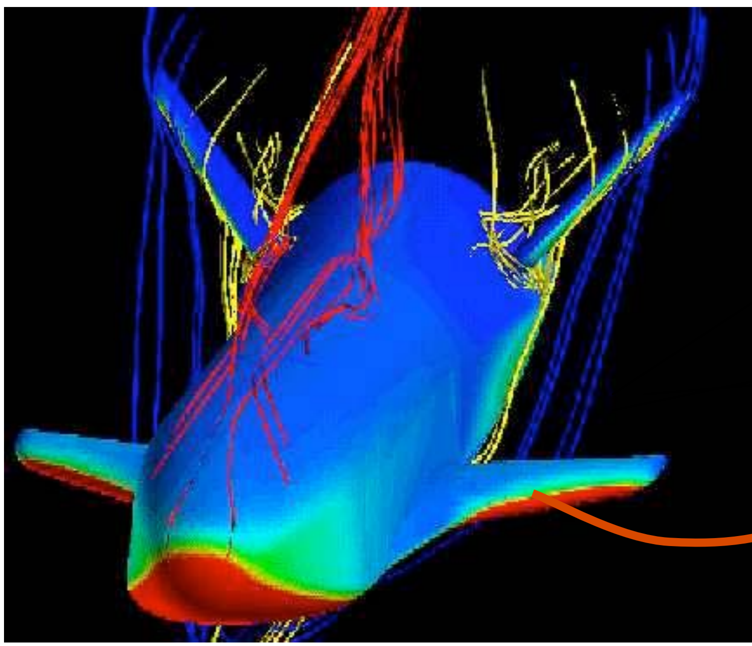
Instead of using analytically defined functions:

- Enrichment functions are produced numerically through a global-local analysis
- Use a *coarse* mesh enriched with global-local functions

- Duarte and Kim, *Computer Methods in Applied Mechanics and Engineering*, 2008.
- O'Hara, Duarte and Eason, *Computer Methods in Applied Mechanics and Engineering*, 2009.

Construction of Global-Local Enrichments

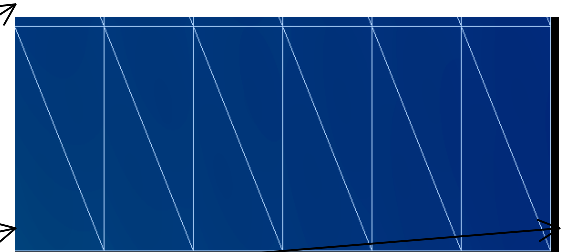
- Step 1: Apply thermo, mechanical and acoustic loads to global structural model. Solve using best available FEM
- Step 2: Extract (automatically) local domains from (coarse) global mesh



Initial global problem

u_G^0 = solution of global problem

Boundary conditions



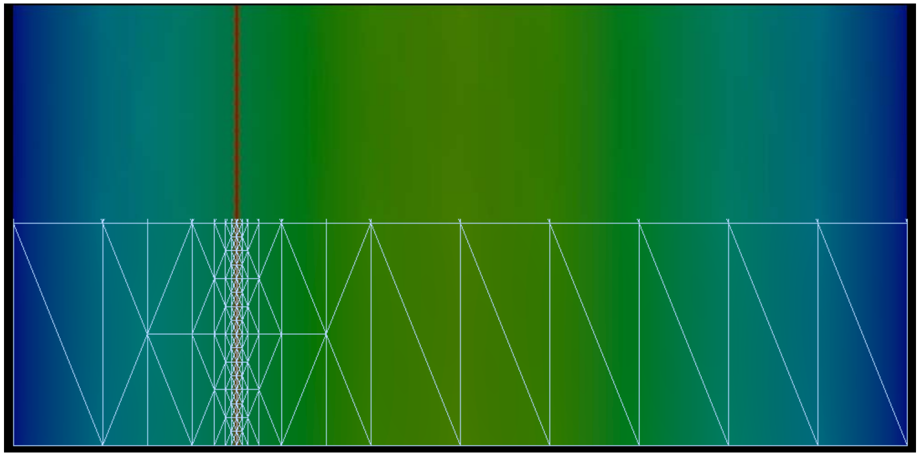
Local problem extracted from global domain

Boundary conditions for local problems provided by global solution:

$$\boxed{u_{loc} = u_G^0 \quad \text{on } \partial\Omega_{loc}}$$

Construction of Global-Local Enrichments

- Step 3: Local problems are solved using *hp* GFEM.
  Use best available closed-form enrichment functions.



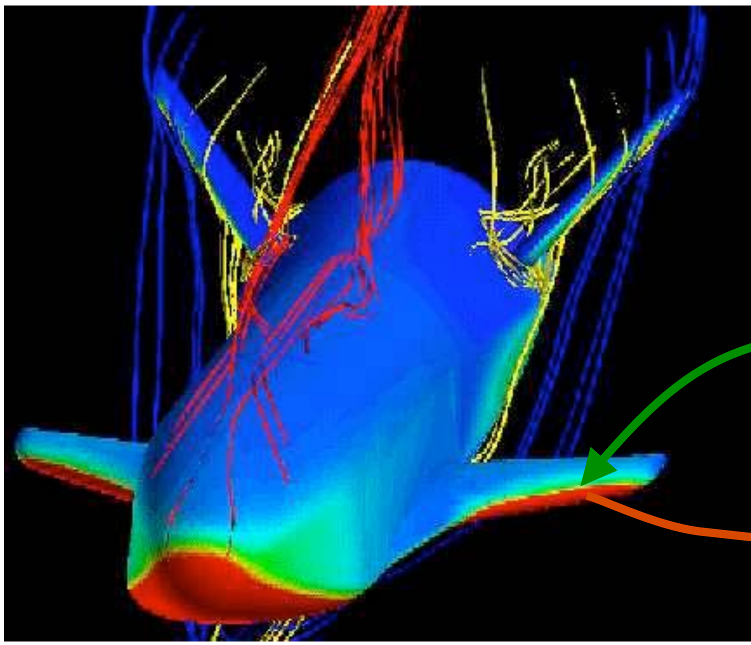
Hp adapted local mesh

u_{loc} = Solution of local problem

- This is just the classical global-local FEM approach
- **Major issue**:
  - Quality of local solutions affected by boundary conditions (global solutions)

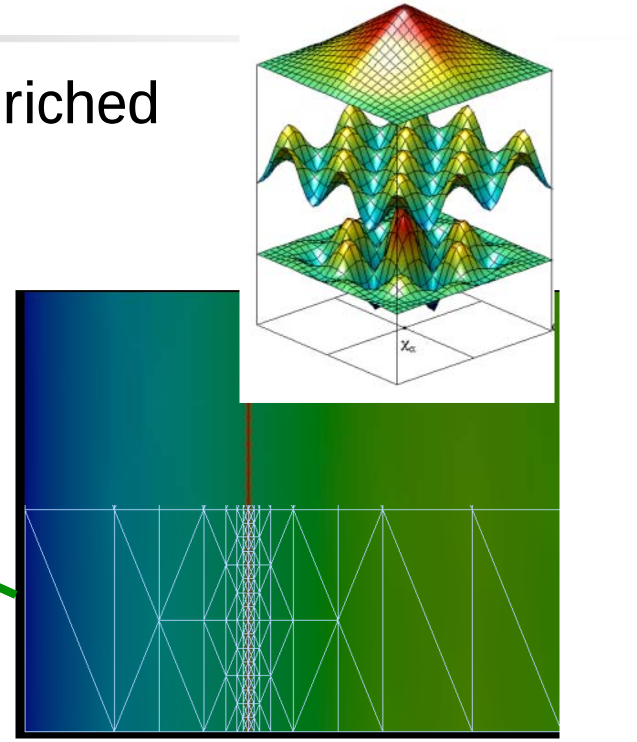
Construction of Global-Local Enrichments

- Defining Step: Global (coarse) mesh is enriched with local solutions



Enrichment functions

Boundary conditions



Hp adapted local mesh

Initial/Enriched global problem

Enrichment of global FEM discretization with local solution:

$$\boxed{\phi_\alpha = \varphi_\alpha u_{loc}}$$

- Only few dofs added to global problem

- Repeat procedure if needed: Update local BCs and enrichment functions

Solution of Enriched Global Problem (Step 4)

$$\boldsymbol{u}_E = \underbrace{\tilde{\boldsymbol{u}}^0}_{\text{coarse scale (FEM)}} + \underbrace{\boldsymbol{u}^{\text{gl}}}_{\text{fine scale (GFEM)}} = \left[\boldsymbol{N}^0 \boldsymbol{N}^{\text{gl}}\right] \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^0 \\ \underline{\boldsymbol{u}}^{\text{gl}} \end{bmatrix}$$

$\underline{\tilde{\boldsymbol{u}}}^0 \;=\;$ DOFs associate with FEM discretization

$\underline{\boldsymbol{u}}^{\text{gl}} \;=\;$ DOFs associate with G-L (hierarchical) enrichments

$$\text{dim}(\underline{\boldsymbol{u}}^{\text{gl}}) << \text{dim}(\underline{\tilde{\boldsymbol{u}}}^0)$$

Use, e.g., static condensation of $\underline{\boldsymbol{u}}^{\text{gl}}$

Strain-displacement matrix is given by

$$\boldsymbol{B}_E = \boldsymbol{L}\left[\boldsymbol{N}^0 \boldsymbol{N}^{\text{gl}}\right] = \left[\boldsymbol{B}^0 \boldsymbol{B}^{\text{gl}}\right]$$

This leads to

$$\begin{bmatrix} \boldsymbol{K}^0 & \boldsymbol{K}^{0,\mathrm{gl}} \\ \boldsymbol{K}^{\mathrm{gl},0} & \boldsymbol{K}^{\mathrm{gl}} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\boldsymbol{u}}}^{\,0} \\ \underline{\boldsymbol{u}}^{\,\mathrm{gl}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^0 \\ \boldsymbol{F}^{\mathrm{gl}} \end{bmatrix}$$

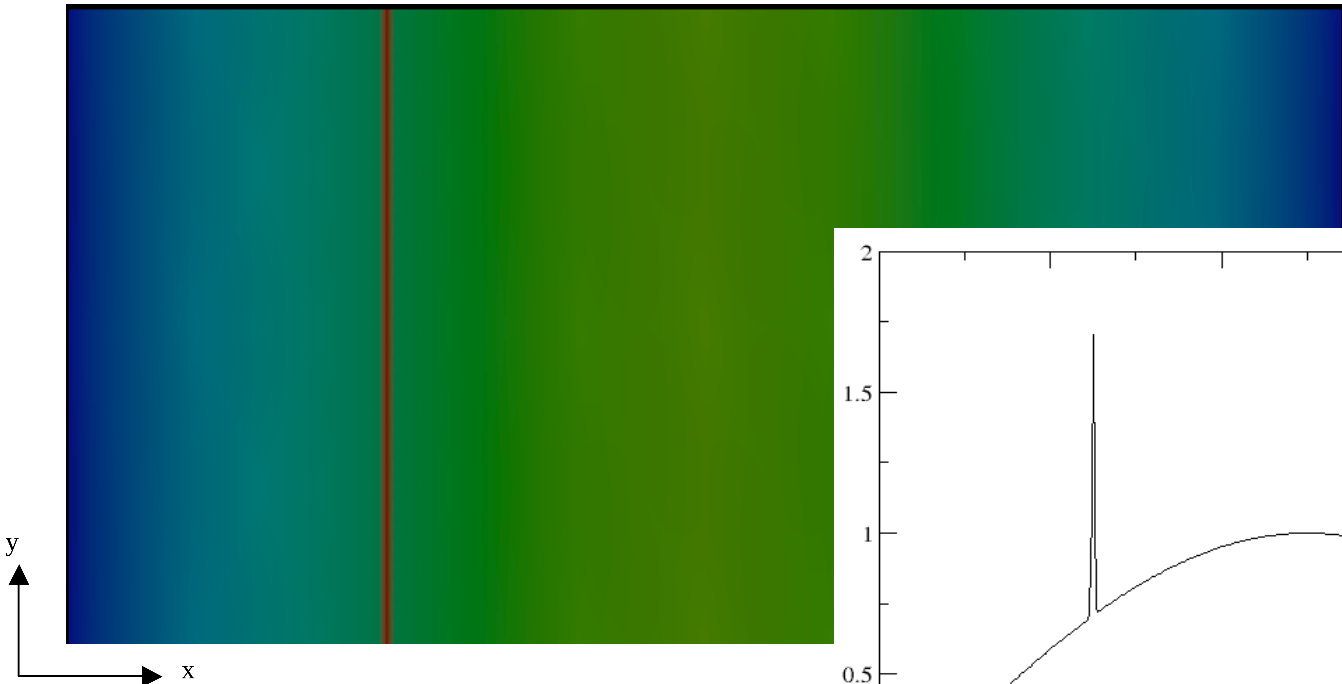
Where

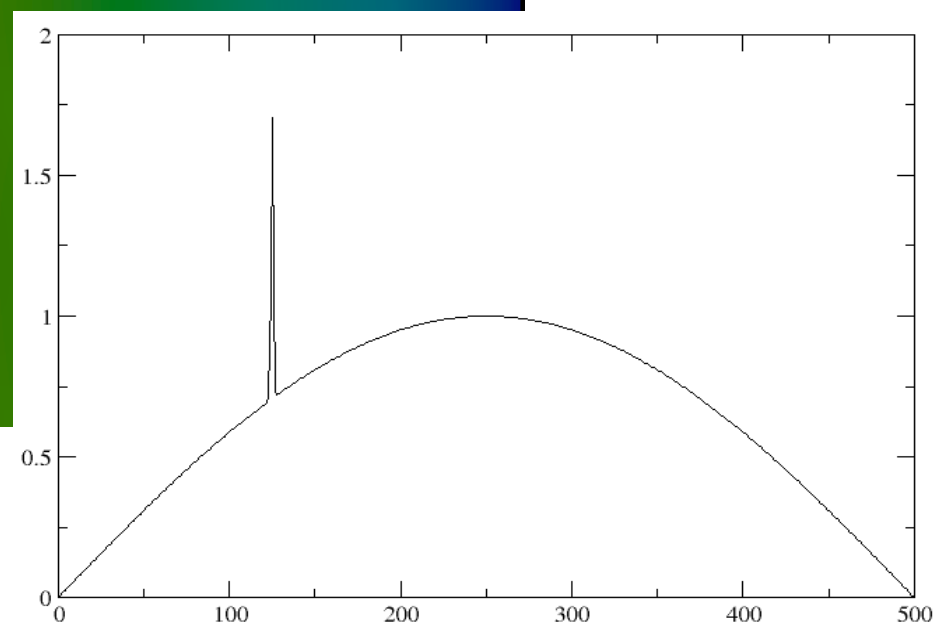
$$\boldsymbol{K}^0 = \int_{\Omega_G} (\boldsymbol{B}^0)^T \boldsymbol{D} \boldsymbol{B}^0 d\Omega$$

$$\boldsymbol{K}^{0,\mathrm{gl}} = \int_{\Omega_{\mathrm{loc}}} (\boldsymbol{B}^0)^T \boldsymbol{D} \boldsymbol{B}^{\mathrm{gl}} d\Omega \qquad \boldsymbol{K}^{\mathrm{gl}} = \int_{\Omega_{\mathrm{loc}}} (\boldsymbol{B}^{\mathrm{gl}})^T \boldsymbol{D} \boldsymbol{B}^{\mathrm{gl}} d\Omega$$

Benchmark Problem

Temperature on a 500 x 250 x 30 mm plate





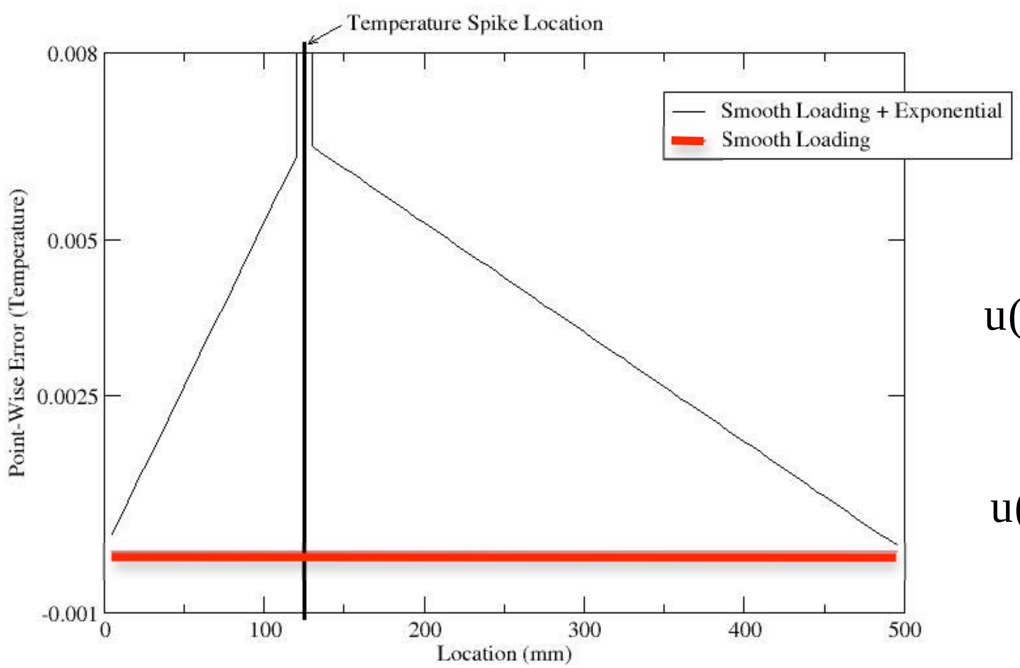
$$u(x) = e^{-\gamma(x-x_0)^2} + \sin\left(\frac{\pi x}{L}\right)$$ Merle & Dolbow (2002)

Steady-State Heat Eq.

$$-\kappa\nabla^2 u = Q(x, y, z) \, in \, \Omega$$

Why bother resolving this local behavior?

- **Error in solution may be large even far from thermal spike (numerical pollution)**

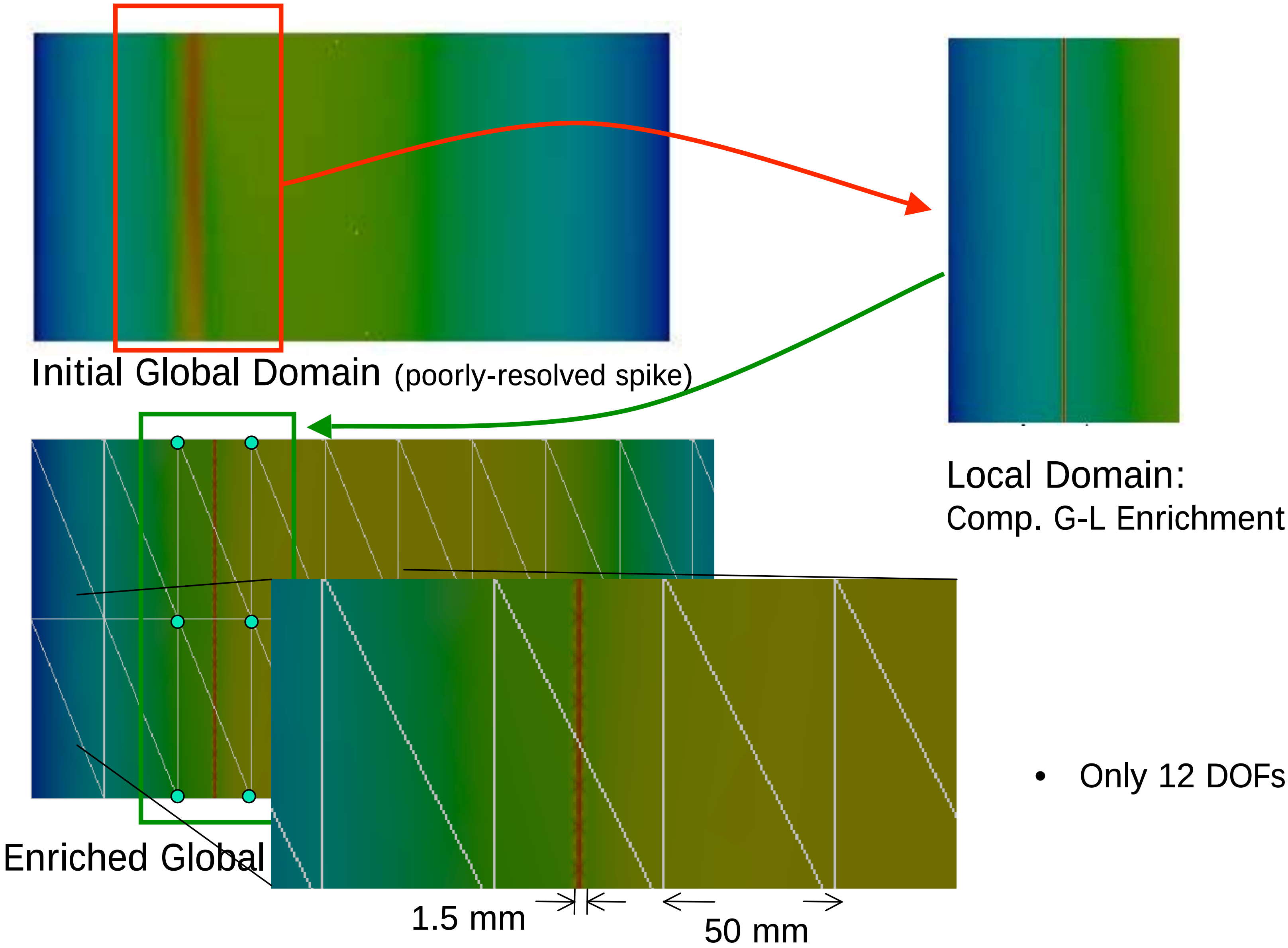


$$u(x) = \sin\left(\frac{\pi x}{L}\right) + e^{-\gamma (x - x_0)^2}$$

$$u(x) = \sin\left(\frac{\pi x}{L}\right)$$

- Can not predict local damage and failure if thermo-mechanical effects are not accurately captured

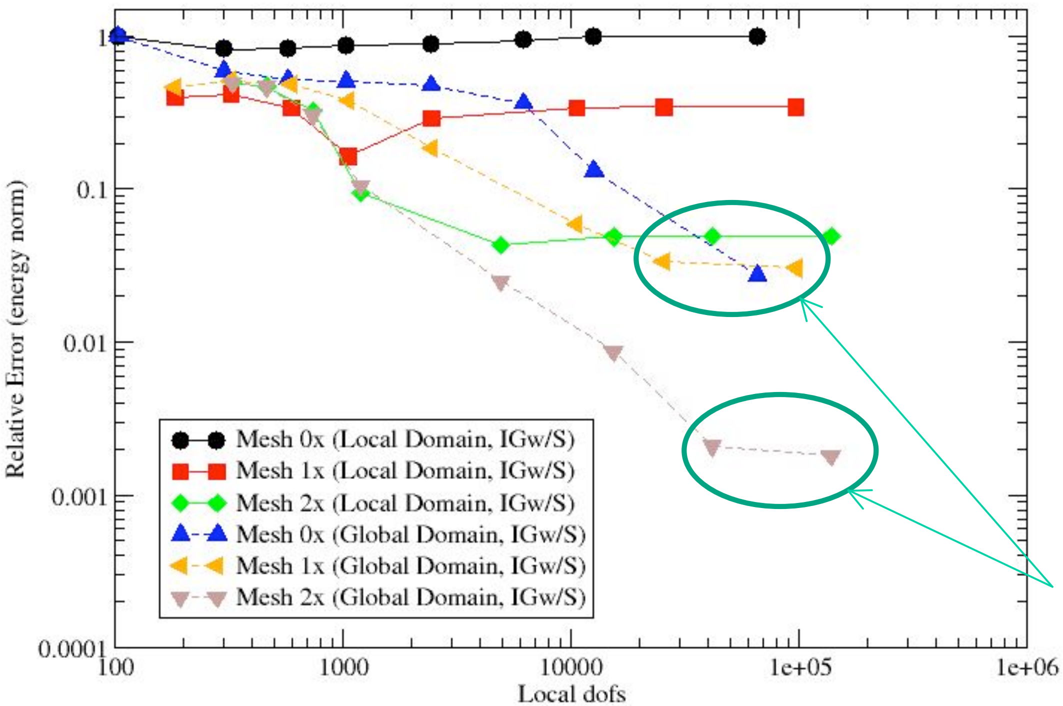
Effectiveness of GFEM^{gl} for Benchmark Problem



- Only 12 DOFs added!

h-Extensions in Local Domains: Effects of BC Quality

- Mesh refine in local domains only



- Three coarse global meshes
  - Mesh 0x - 10 elements in x-dir
  - Mesh 1x - 20 elements in x-dir
  - Mesh 2x - 40 elements in x-dir
- Poor or no convergence in local domain
- Improved convergence in global domain
- Limit to range of convergence in enriched global domain for Mesh 1x and Mesh 2x

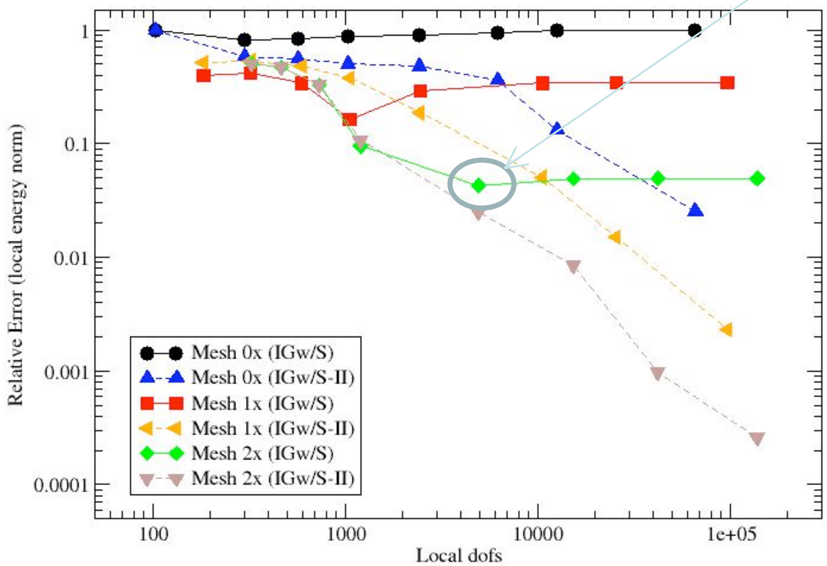
Local Domain
Enriched Global Domain

Two-Step Approach to Improve BC's

Perform second iteration, i.e. use enriched global solution as initial global step for a second global-local iteration; proceed as usual

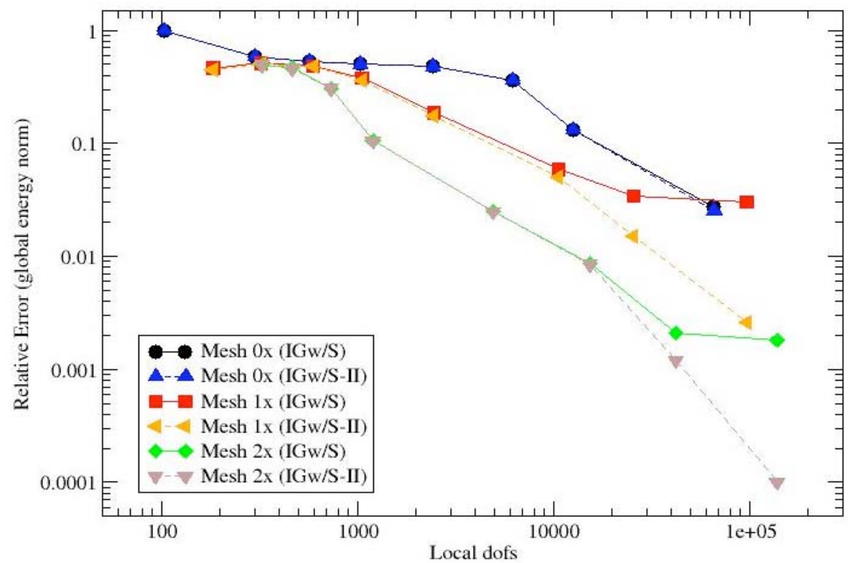
Look more closely at this local refinement level

Local Domain



Convergence obtained in local domain.

Enriched Global Domain



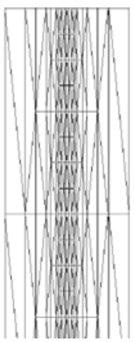
Extended range of convergence in global domain.

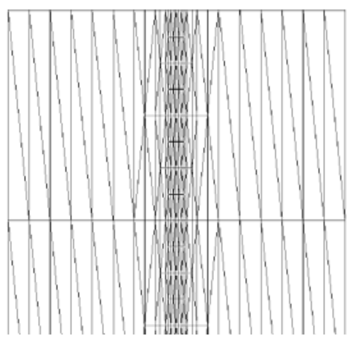
——— Iteration 1

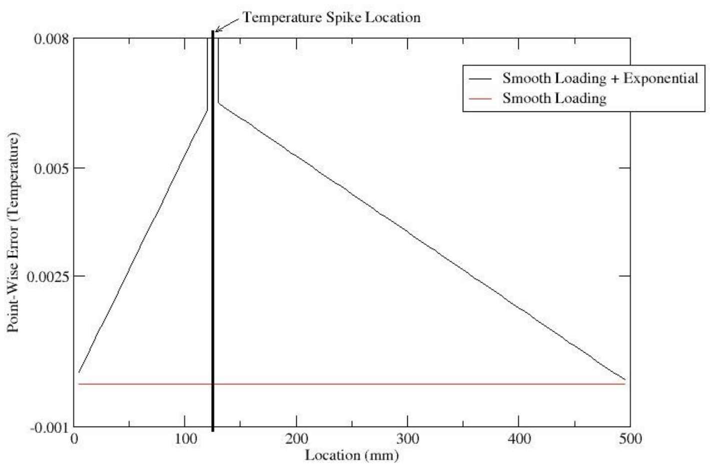
---------- Iteration 2

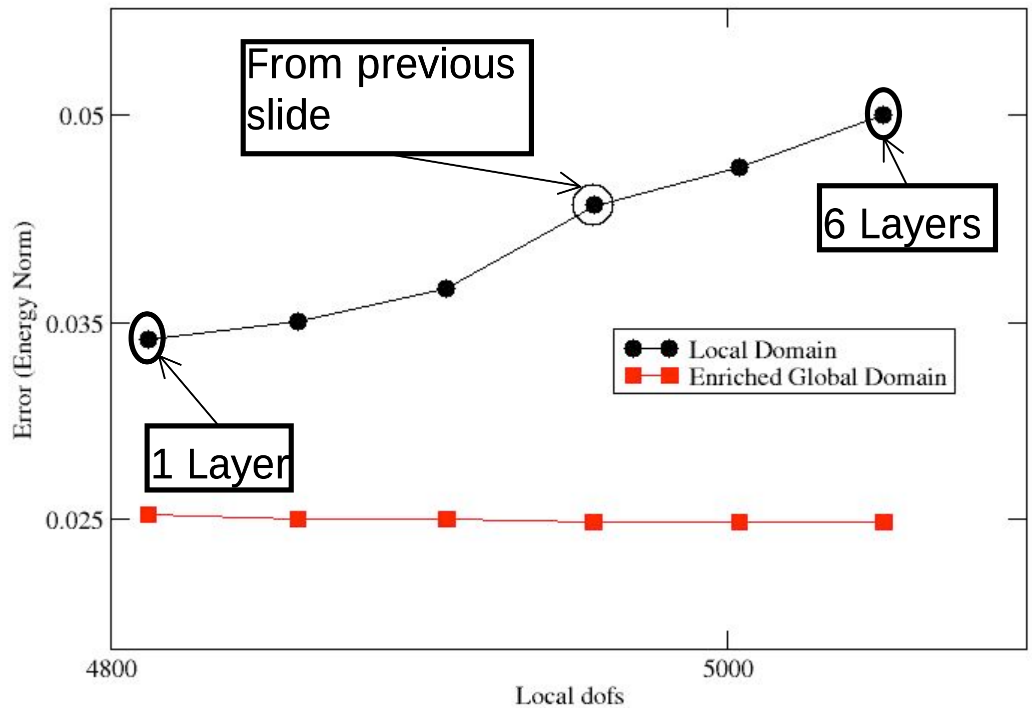
Effect of Local Domain Size

Keeping local refinement fixed, increase size of local domain to improve BCs. Commonly used in the Global-Local FEM.



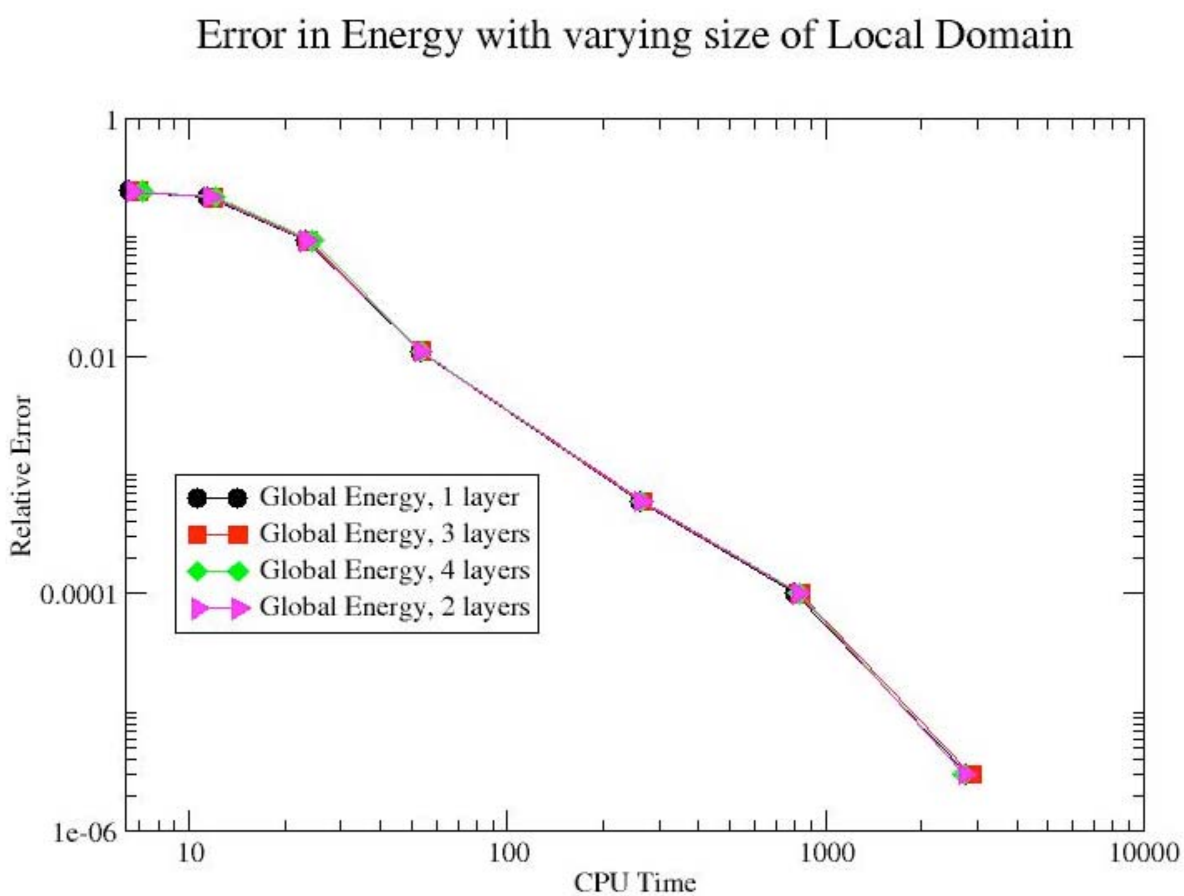






- Numerical pollution generates very poor BCs across domain.
- Local solution does not improve with increasing size of local domain.

Effect of Size of Local Domain on Enriched Global Solution



Size of local domain has little effect on the enriched global solution!

Computational Performance of GFEM^{gl}

- Multiple-site thermal analysis of a stiffened plate

✓ Panel subjected to sharp fluxes over small regions plus a smooth thermal load

✓ Thermal conductivity:

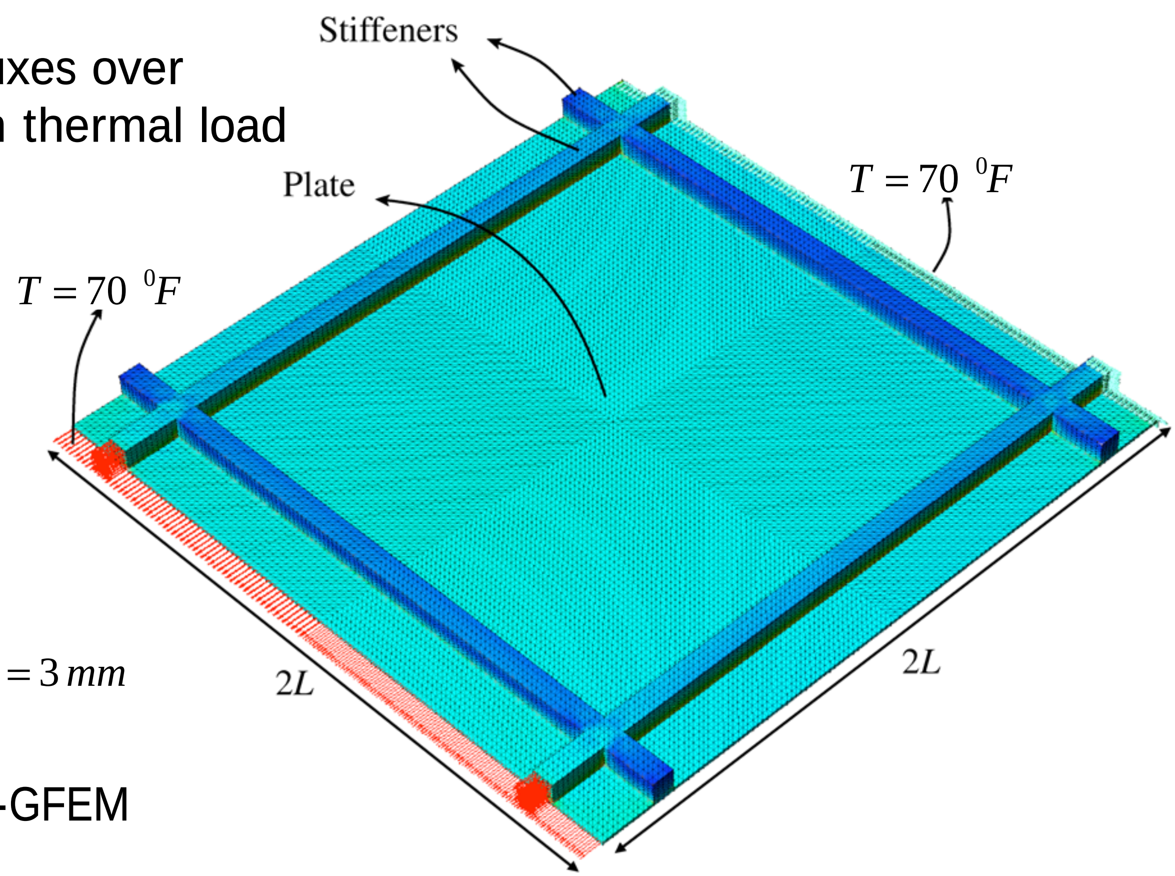
$$\kappa = 5.5 \, {}^{W}\!/\!{}_{mm^0F}$$

✓ Dimensions

$L = 300\,mm$ $\quad t = 3\,mm$

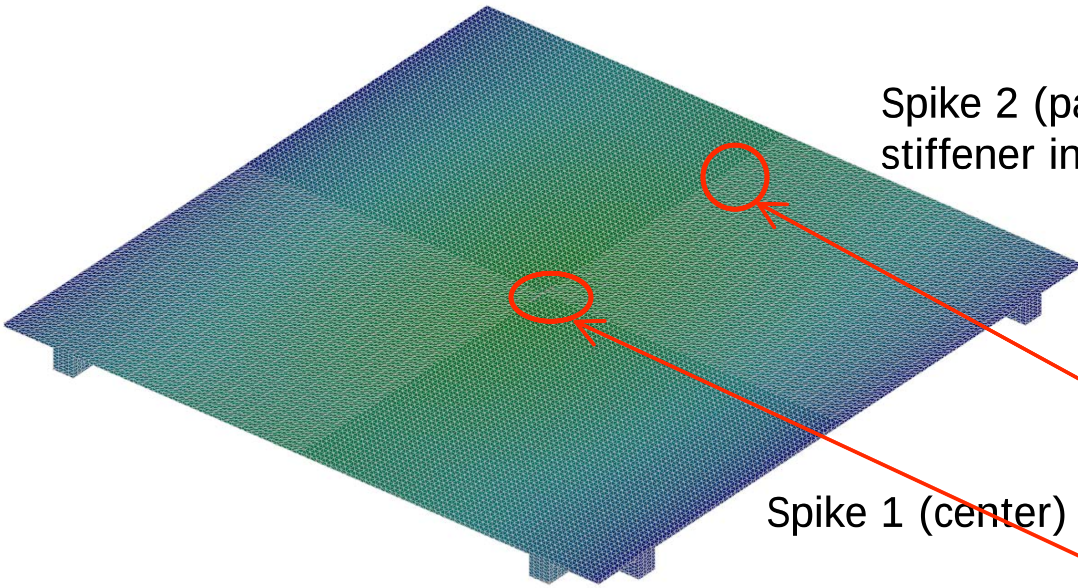
$b = 20\,mm$ $\quad d = 50\,mm$ $\quad h_e = 3\,mm$

✓ Reference solution from *hp*-GFEM

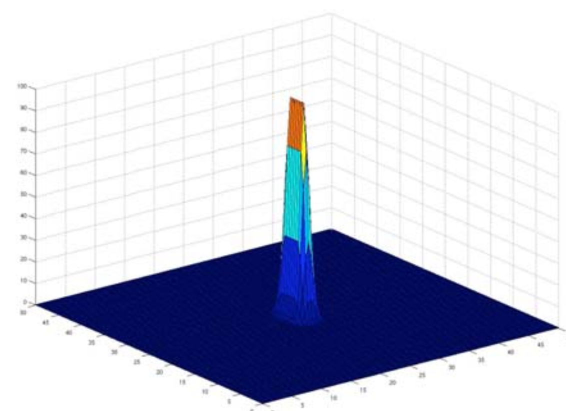


Computational Performance

- Multiple-site thermal analysis – Three spike configurations



Solution of initial global problem without thermal spikes

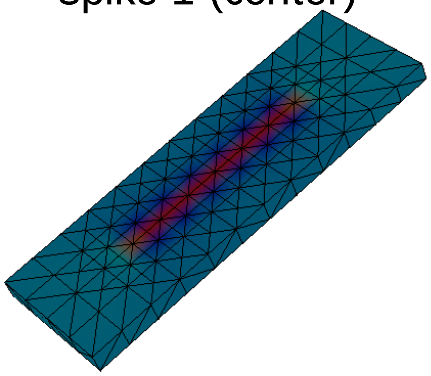


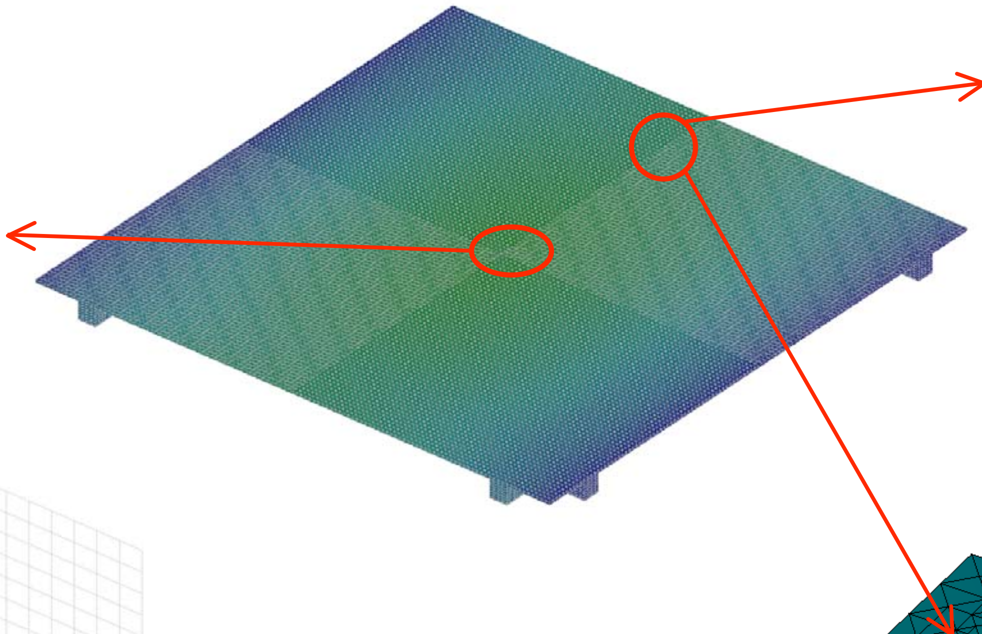
- GFEM/FEM must solve each problem from scratch
- GFEMgl can re-use initial global solution (hierarchical enrich):
  - Just add proper global-local enrichment to global problem

Computational Performance

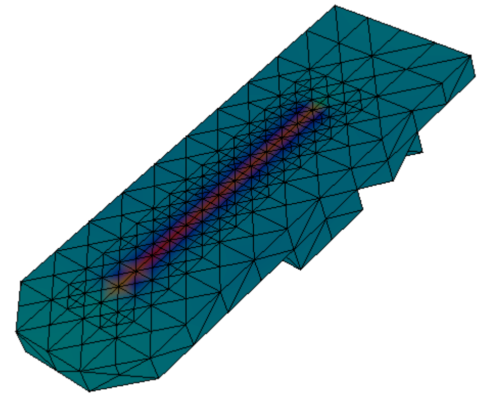
- Compute global-local enrichments on the fly

Local prob. for
Spike 1 (center)

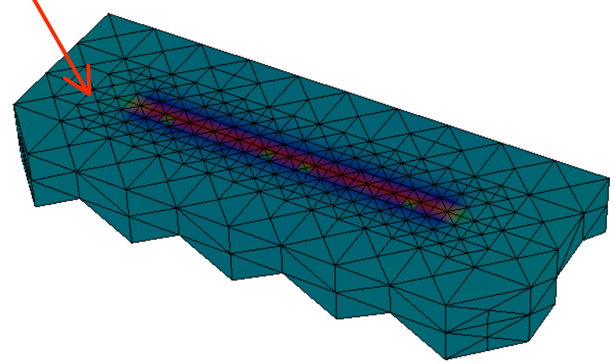


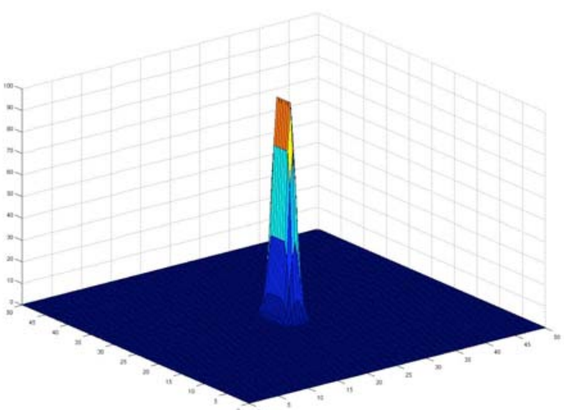


Spike 2 (panel,
stiffener interface)



Spike 3
(over stiffener)

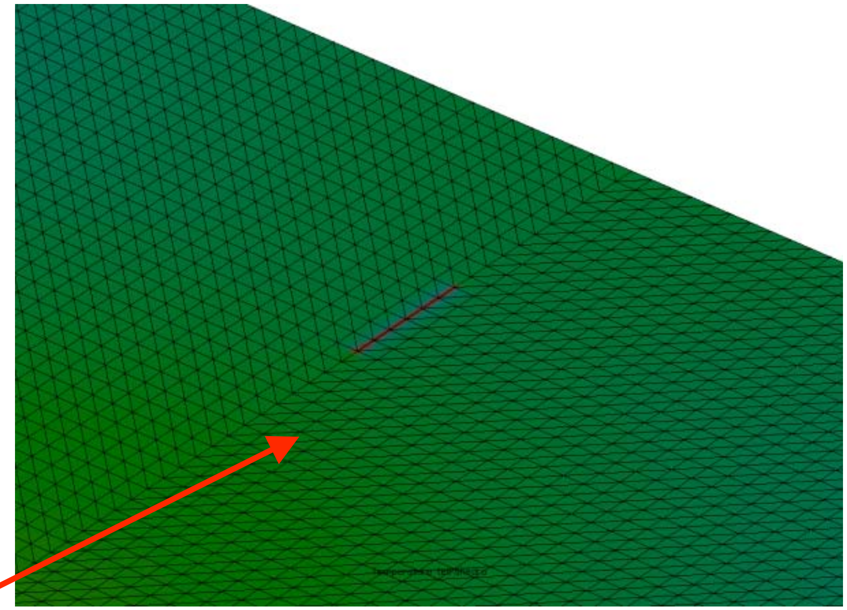


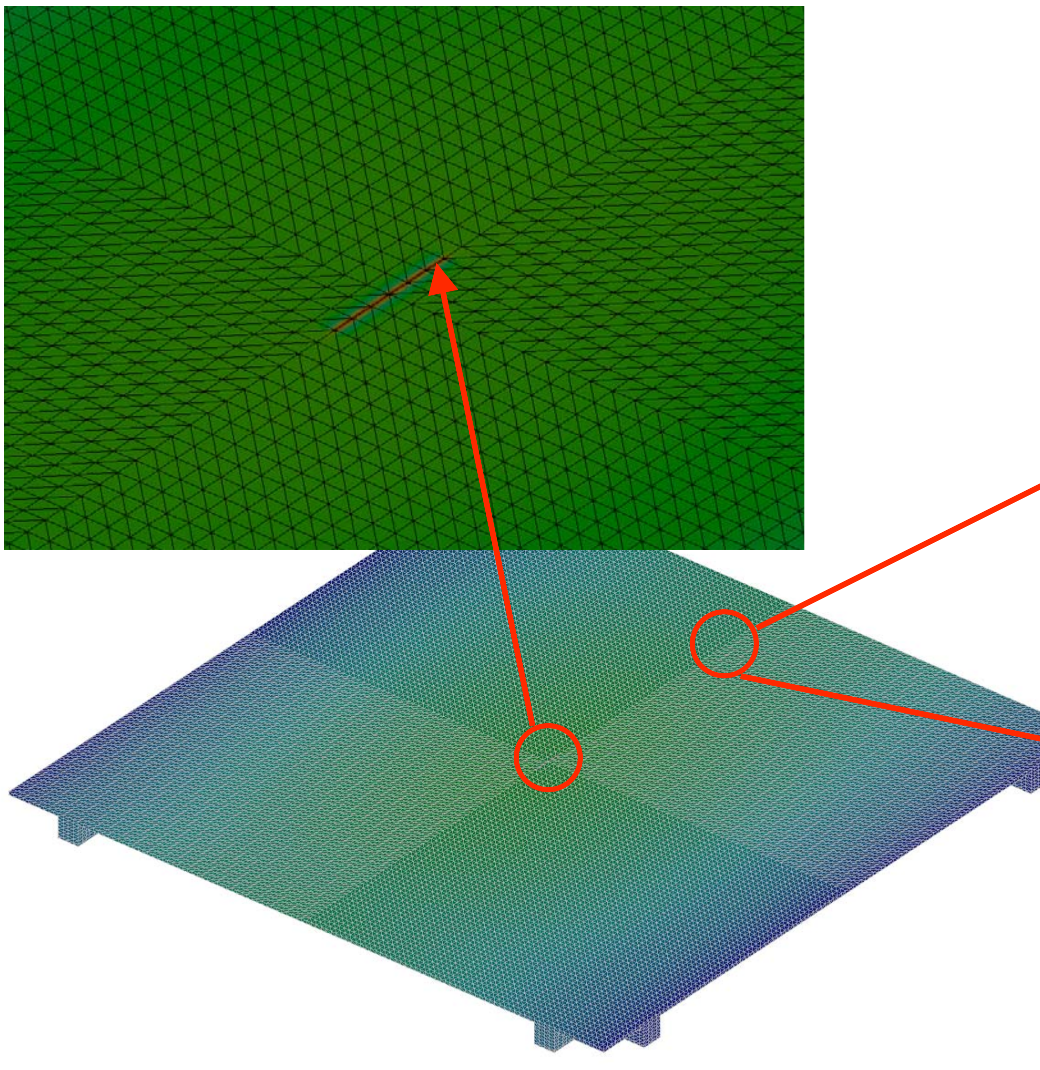


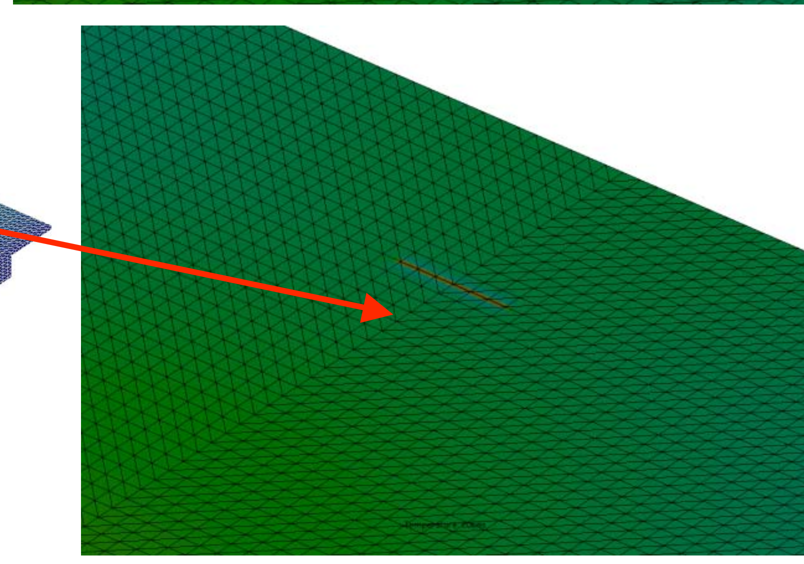
Prescribed flux

GFEM^{gl} Solutions

- <u>Same</u> global mesh used for all three cases



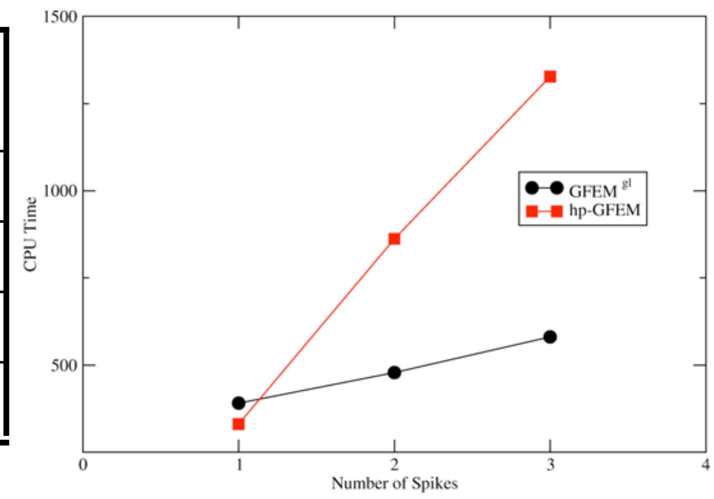




Computational Performance

- Hp-GFEM/FEM performance

Spike	Num DOFs	CPU time (s)	Internal energy
1	203,430	331	1.565e7
2	241,492	531	1.564e7
3	221,502	465	1.554e7
TOTAL:		**1,327**	



- $GFEM^{g\text{-}l}$ performance

	Num DOFs			CPU time (s)				Internal energy
Spike	Init Gl	Local	Enrich Gl	Init Gl	Local	Enrich Gl	TOTAL	
1		12,930	182,196		10	61	71	1.558e7
2	182,136	22,220	182,205	319	36	52	88	1.546e7
3		29,890	182,186		50	57	107	1.555e7
TOTAL:							**585**	

Transient Heat Conduction

Strong Form of Governing Equation:

$$\rho c \frac{\partial u}{\partial t} - \kappa \nabla^2 u = Q(x, y, z) \, in \, \Omega$$

Weak Form of Governing Equations:

$$[M]\{v^{n+1}\} + [K]\{d^{n+1}\} = \{F^{n+1}\}$$

Define the above quantities as:

$$M_{ij}^{el} = \int_{\Omega^{el}} c \phi_i \phi_j d\Omega$$ (Capacity Matrix)

$$K_{ij}^{el} = \int_{\Omega^{el}} \kappa_{pq} \frac{\partial \phi_i}{\partial \phi_p} \frac{\partial \phi_j}{\partial \phi_q} d\Omega$$ (Stiffness Matrix)

$$F_i = \int_{\Omega^{el}} Q(\vec{x}, t) \phi_i d\Omega + \int_{\Gamma_f^{el}} \bar{f}(\vec{x}, t) \phi_i d\Gamma_f$$ (Load Vector)

- Time Integration Scheme: a-Method

Increments in solution/derivative:

$$d^{n+1} = d^n + \Delta t v^{n+\alpha} \qquad 0 < \alpha \le 1$$

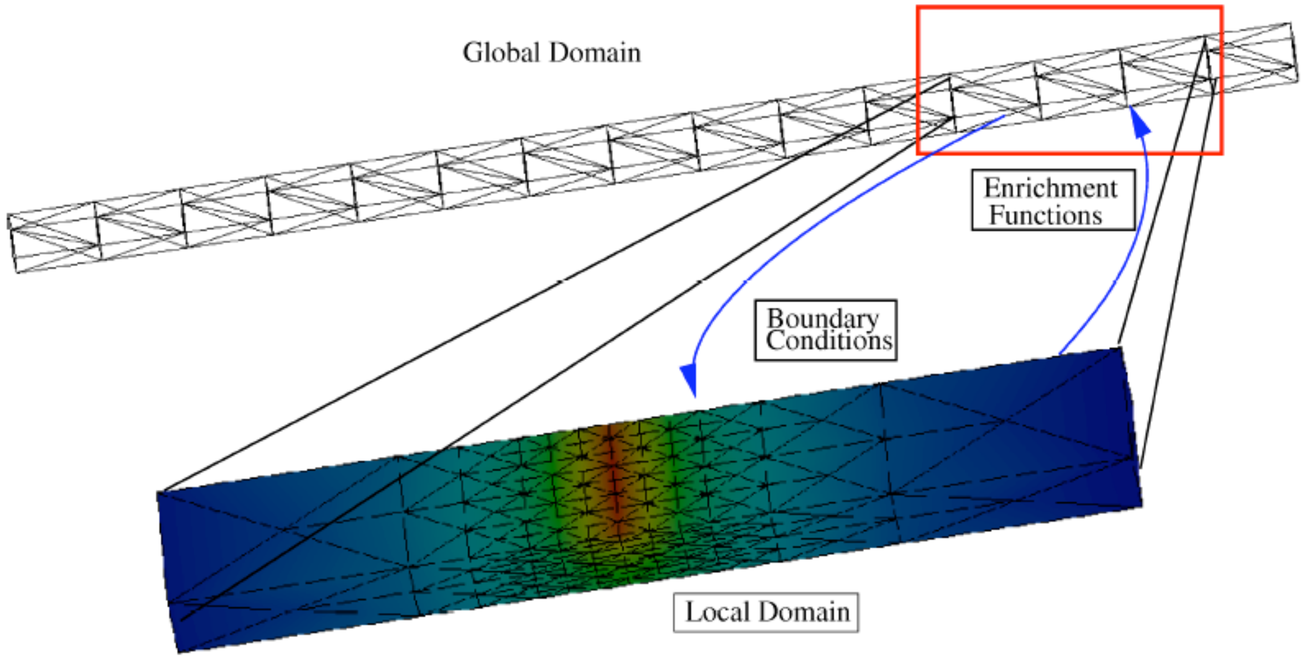
$$v^{n+\alpha} = (1-\alpha) v^n + \alpha v^{n+1}$$

Solve for d^{n+1} and v^{n+1} at each time step using:

$$\frac{1}{\alpha \Delta t}[M + \alpha \Delta t K] d^{n+1} = F^{n+1} + \frac{M}{\alpha \Delta t}\left(d^n + (1-\alpha)\Delta t v^n\right)$$

$$v^{n+1} = \frac{1}{\alpha \Delta t}\left(d^{n+1} - d^n\right) - \frac{1-\alpha}{\alpha} v^n$$

GFEM^{gl} for Time-Dependent Problems



Process repeated at each time-step

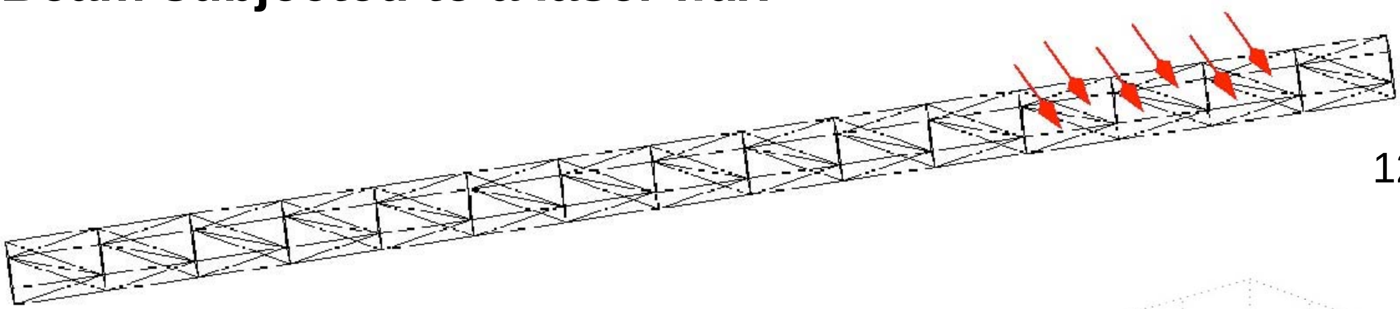
Same basic concept as in steady-state plus:

- boundary conditions taken from enriched global solution at t^n
- enrichment functions generated for enriched global solution at t^{n+1}
- no transient effects considered in local domain

- Use available information to build solution space for next time step

Analysis of Beam Subjected to Laser Heating

- **Beam subjected to a laser flux**



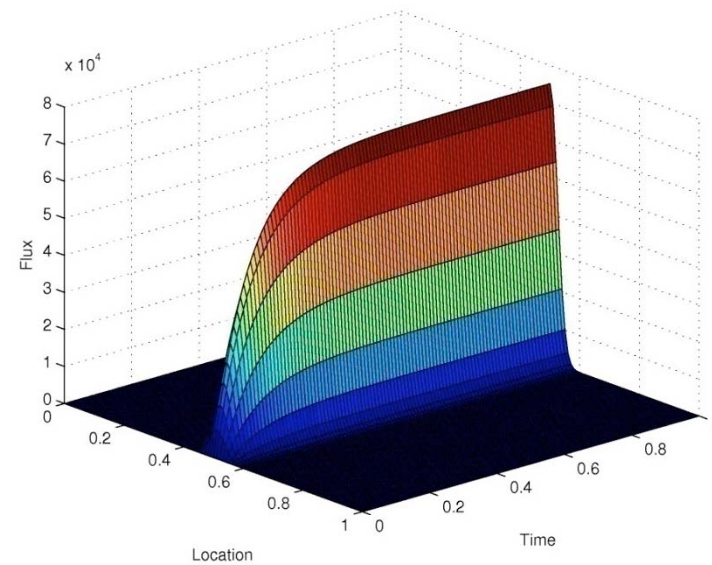
12” x 0.5” x 0.24”

Flux function defined as:

$$q = I_0 * f(t) * \frac{1}{2\pi a^2} * G(x, b, a)$$

$$f(t) = 1 - \exp(-\gamma * t)$$

$$G(x, b, a) = \exp\left(\frac{-(x-b)^2}{2a^2}\right)$$

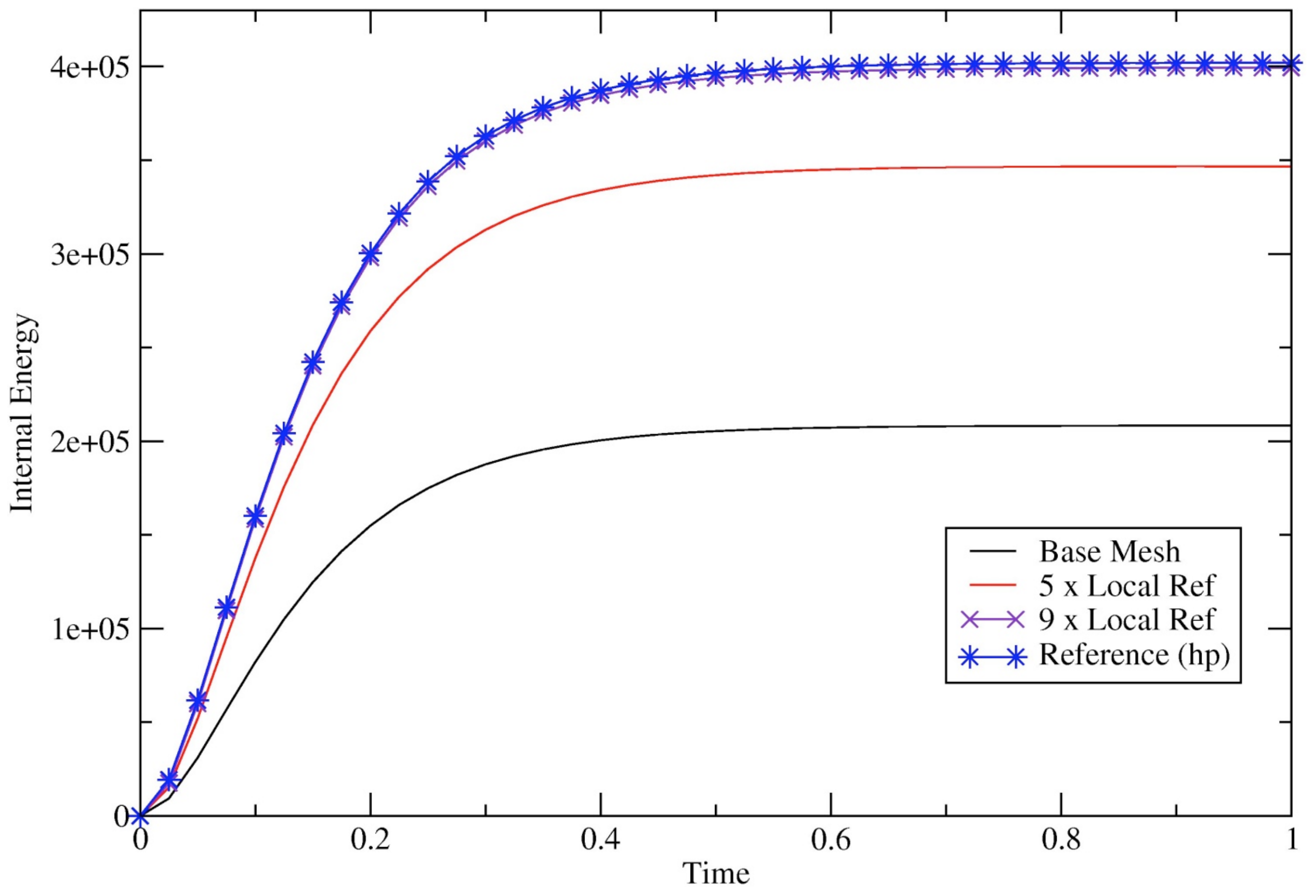


With constants:

I_0=295, a = 0.025, γ = 10.0, b = 9.3

Analysis of Beam Subjected to Laser Heating

- Internal energy of enriched global problem



Simulations:

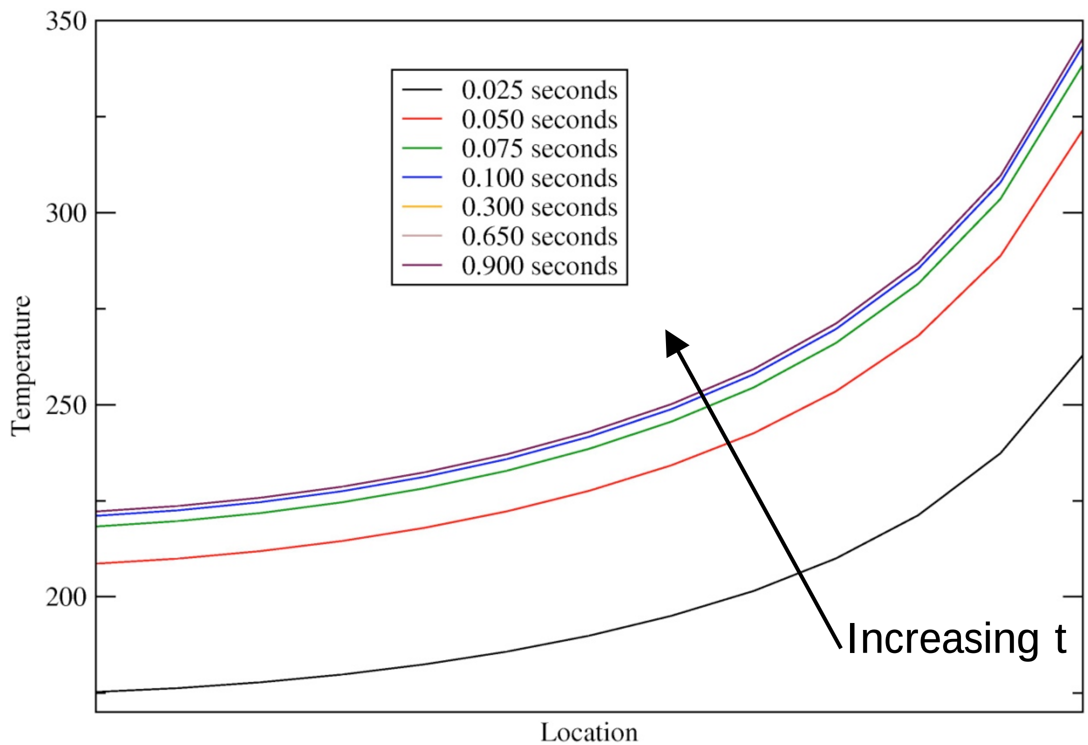
Base Mesh

GFEM^{gl}

Global mesh with h_x = 1.0”

Through-the-Thickness Temperature Distribution

- Enriched global problem

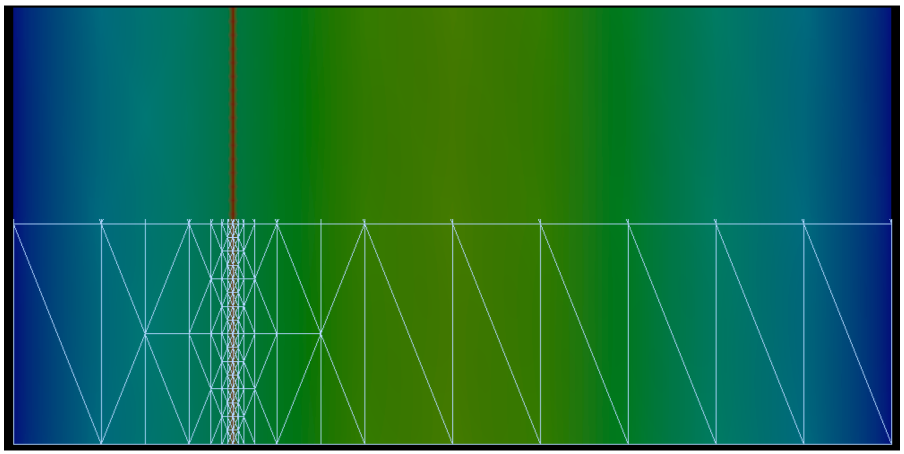


$(\gamma = 50)$

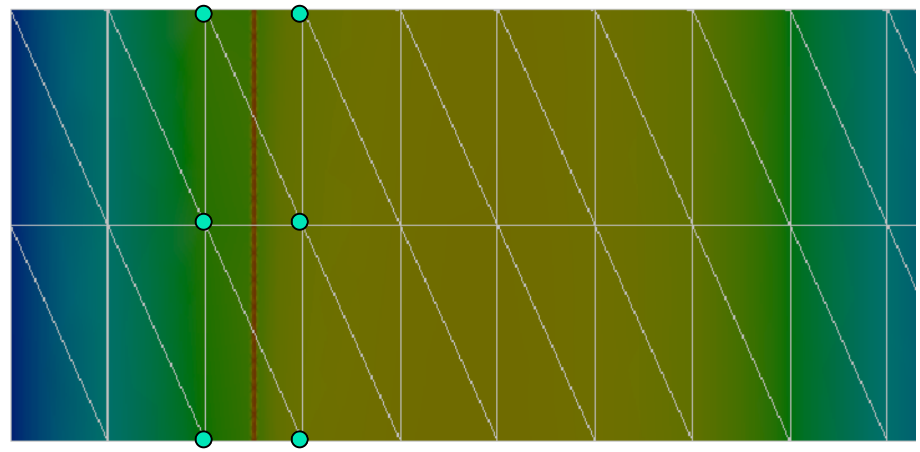
Non-linear through-the-thickness temperature variations

Multi-Scale Problems: Computational Challenges

Adaptive FEM

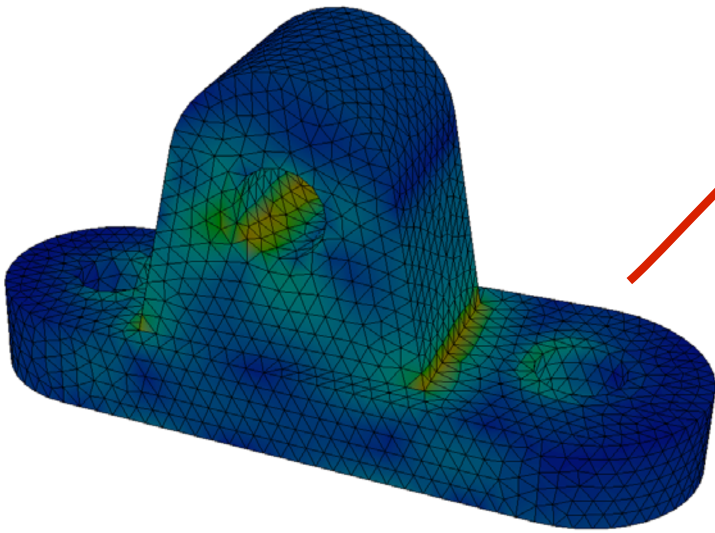


Adaptive GFEM^{gl}



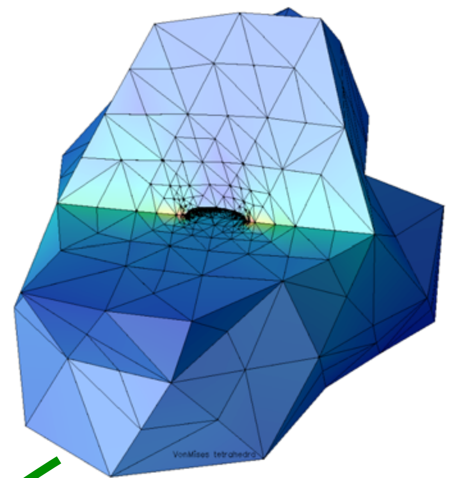
- Adaptive mesh refinement (AMR) is required in standard FE Methods
- The GFEM^{gl} can deliver accurate solutions on coarse meshes
- Most fine scale effects are non-local (pollution effect)
  - ➢ Standard global-local analysis is not robust in general
- GFEM^{gl} : Account for interactions among non-separable scales
  - ➢ More robust than global-local FEM at a comparable computational cost

Application to 3-D Fracture



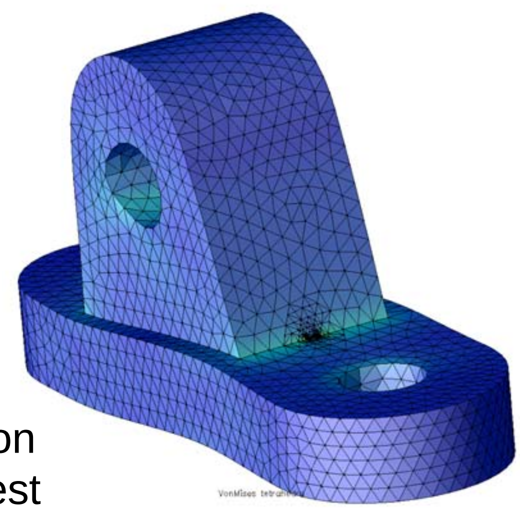
Global solution
BC for local
prob.

Enrichment Fns



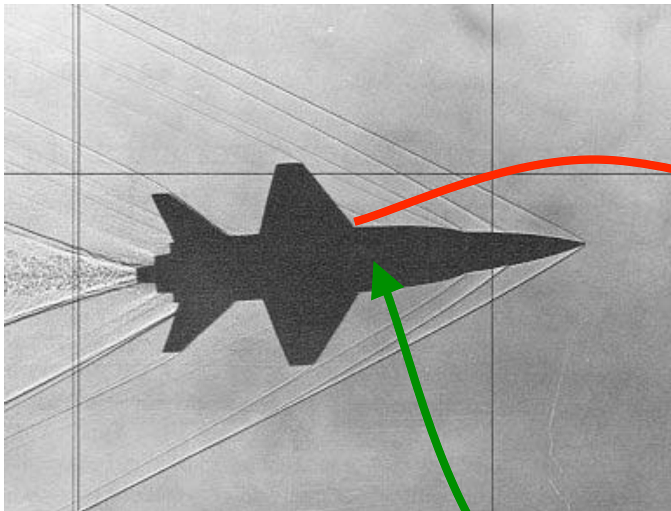
Local solution
Global-local enrichment

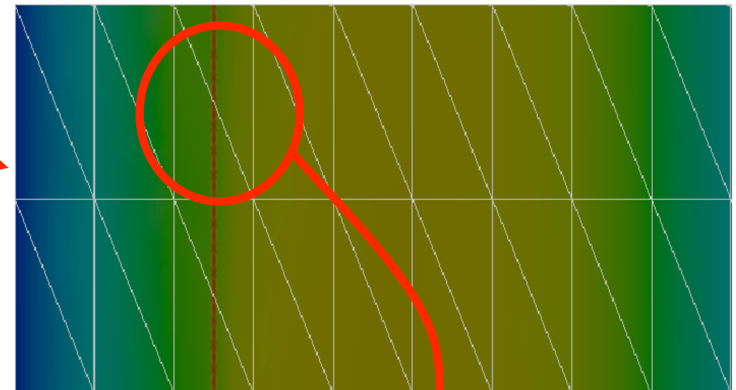
Initial global solution
Uncracked domain

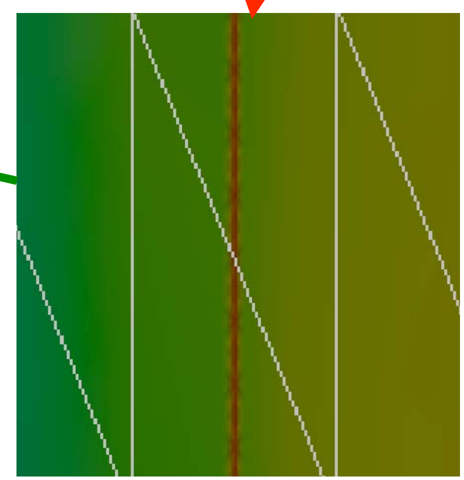


Enriched global solution
Get quantities of interest

Questions ?







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